# 公式集

TKG

2025 年 6 月 1 日

## 目 次



## 1　INTRODUCTION

学生のときにこういう方面に凄くはまってた時期がありました。その時期に導出した公式とかをノートにまとめたものです。 あまり厳密なことは気にせずにやってたのであしからずです。

## 2　積分表示された関数の級数表示の 1 例

様々な積分表示された関数を級数で表すこと。

$$\begin{aligned}
\int_0^1 \frac{(1-x)^{s-1}}{(1-ux)^{\alpha+1}} x^{\beta-1} dx &= \int_0^1 (1-x)^{s-1} x^{\beta-1} \sum_{n=0}^{\infty} \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} u^n x^n dx \\
&= \sum_{n=0}^{\infty} \frac{(\alpha+1)\cdots(\alpha+n)}{n!} u^n \int_0^1 (1-x)^{s-1} x^{\beta+n-1} dx \\
&= \sum_{n=0}^{\infty} \frac{(\alpha+1)\cdots(\alpha+n)}{n!} \frac{\beta(\beta+1)\cdots(\beta+n-1)}{(s+\beta)(s+\beta+1)\cdots(s+\beta+n-1)} u^n \frac{\Gamma(s)\Gamma(\beta)}{\Gamma(s+\beta)} \qquad (1)
\end{aligned}$$

これに $u \to 1$ の極限を取ればまとめると

$$\frac{\Gamma(s+\beta)\Gamma(s-\alpha)}{\Gamma(s)\Gamma(s-\alpha+\beta)} = 1 + \frac{\alpha\beta}{1!(s+\beta)} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!(s+\beta)(s+\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!(s+\beta)(s+\beta+1)(s+\beta+2)} + \cdots \qquad (2)$$

を得る。同様な変形をして

$$\int_0^1 x^{s-1}(1-ux)^{\beta-1}dx = \frac{1}{s} - \frac{\beta-1}{1!(s+1)}u + \frac{(\beta-1)(\beta-2)}{2!(s+2)}u^2 - \frac{(\beta-1)(\beta-2)(\beta-3)}{3!(s+3)}u^3 + \cdots \tag{3}$$

よって $u \to 1$ の極限を取れば

$$\frac{\Gamma(s)\Gamma(\beta)}{\Gamma(s+\beta)} = \frac{1}{s} - \frac{(\beta-1)}{1!(s+1)} + \frac{(\beta-1)(\beta-2)}{2!(s+2)} - \frac{(\beta-1)(\beta-2)(\beta-3)}{3!(s+3)} + \cdots \tag{4}$$

などを得る。例えば $s=\beta=\frac{\pi}{2}$ を代入して

$$\pi = 2 + \frac{1}{1!\cdot 3} + \frac{1\cdot 3}{2!\cdot 5\cdot 2} + \frac{1\cdot 3\cdot 5}{3!\cdot 7\cdot 2^2} + \frac{1\cdot 3\cdot 5\cdot 7}{4!\cdot 9\cdot 2^3} + \cdots \tag{5}$$

などが得られる。また $\beta = 1-s$ を代入すれば

$$\frac{\pi}{\sin\pi s} = \frac{1}{s} + \frac{s}{1!(s+1)} + \frac{s(s+1)}{2!(s+2)} + \frac{s(s+1)(s+2)}{3!(s+3)} + \cdots \tag{6}$$

を得る。etc...

## 3 オイラー・マクローリンの公式

関数 $F(x+h)$ のテイラー展開は形式的には

$$\begin{aligned} F(x+h) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d}{dx}\right)^n F(x) \\ &= \exp\left(h\frac{d}{dx}\right)F(x) \end{aligned} \tag{7}$$

と書ける。このことに留意して

$$\begin{aligned} F(x)+F(x+h)+F(x+2h)+\cdots+F(x+Nh) &= \sum_{n=0}^{N} e^{nh\frac{d}{dx}}F(x) \\ &= \frac{1-e^{(N+1)h\frac{d}{dx}}}{1-e^{h\frac{d}{dx}}}F(x) \end{aligned} \tag{8}$$

と形式的に書ける。ここで"複素解析のお話"note によれば

$$\frac{x}{1-e^x} = -1 + \frac{1}{2}x - \frac{B_1}{2!}x^2 + \frac{B_2}{4!}x^4 - \frac{B_3}{6!}x^6 + \cdots \tag{9}$$

とかけた。従ってこの式に形式的に $x$ に $\frac{d}{dx}$ を"代入"すると、$\frac{dF}{dx}(x) = f(x)$ と置けば

$$\begin{aligned} \sum_{n=0}^{N} f(x+nh) &= \int_x^{x+(N+1)h} f(y)dy - \frac{1}{2}\Big(f(x+(N+1)h)-f(x)\Big) + \frac{B_1}{2!}\frac{d}{dx}\Big(f(x+(N+1)h)-f(x)\Big) \\ &\quad - \frac{B_2}{4!}\frac{d^3}{dx^3}\Big(f(x+(N+1)h)-f(x)\Big) + \frac{B_3}{6!}\frac{d^5}{dx^5}\Big(f(x+(N+1)h)-f(x)\Big) - \cdots \end{aligned} \tag{10}$$

となる。これをオイラー・マクローリンの公式という。これを用いて簡単な例を計算してみる。

まずオイラー・マクローリンの公式を用いれば

$$S^k(N) := \sum_{n=1}^{N} n^k \tag{11}$$

が計算できる。$f(x)=x^k$ と置き、最後に $x=0$、$h=1$ と置けばよい。計算すると

$$S^k(N) = \frac{(N+1)^{k+1}}{k+1} - \frac{1}{2}(N+1)^k + \frac{B_1}{2!}k(N+1)^{k-1} - \frac{B_2}{4!}k(k-1)(k-2)(N+1)^{k-3} + \cdots \tag{12}$$

となる。ここで和は $N+1$ の 1 次以上の項までとる。$k=1$ の時は

$$S^1(N) = \frac{N(N+1)}{2} \tag{13}$$

$k=2$ の時は

$$S^2(N) = \frac{N(N+1)(2N+1)}{6} \tag{14}$$

などとよく知られた公式が得られる。

他にも応用例はあるが、ここでは省略する。また工夫すれば $\sum_n (-1)^n f(n)$ のようなものも同様な計算をして公式を得ることが出来るが、それも省略する。

## 4 冪級数表示の 1 例

"解析の話発展編"note によれば

$$\int_0^\infty f(x)x^{s-1}dx = \Gamma(s)\phi(s) \tag{15}$$

の時

$$f(x) = \sum_{n=0}^\infty (-1)^n \frac{\phi(-n)}{\Gamma(1+n)} x^n \tag{16}$$

となる。この関係式を使って得られるいくつかの公式

$$\begin{aligned} \sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x \cdots &= \frac{1}{2}\cot\frac{x}{2} \\ \sin x - \sin 2x + \sin 3x - \sin 4x + \cdots &= \frac{B_1}{2!}(2^2-1)x + \frac{B_2}{4!}(2^4-1)x^3 + \frac{B_3}{6!}(2^6-1)x^5 + \cdots \\ &= \frac{1}{2}\tan\frac{x}{2} \end{aligned} \tag{17}$$

他には

$$\frac{e^x}{(e^x-1)^2} = \frac{1}{x^2} - \frac{1}{2}\frac{B_1}{0!} + \frac{1}{4}\frac{B_2}{2!}x^2 - \frac{1}{6}\frac{B_3}{4!}x^4 + \cdots \tag{18}$$

など得られる。これは実際に $\frac{d}{dx}(\frac{1}{e^x-1}) = \frac{-e^x}{(e^x-1)^2}$ を用いて計算した結果と一致する。この調子で計算した結果を列挙していく。

$$
\begin{aligned}
&\sin x+\frac{1}{2^a}\sin 2x+\frac{1}{3^a}\sin 3x+\frac{1}{4^a}\sin 4x+\cdots\\
&=\frac{\pi}{2}-\frac{1}{2}x \qquad (a=1)\\
&=\pi^2 B_1 x-\frac{\pi}{4}x^2+\frac{1}{3!\cdot 2}x^3 \qquad (a=3)\\
&=\frac{2^3\pi^4}{4!}B_2 x-\frac{\pi^2}{3!\cdot 6}x^3+\frac{\pi}{4!\cdot 2}x^4-\frac{1}{2\cdot 5!}x^5 \qquad (a=5)\\
&=\frac{2^5\pi^6}{6!}B_3 x-\frac{2^3\pi^4}{3!\cdot 4!}B_2 x^3+\frac{2\pi^2}{5!\cdot 2!}B_1 x^5-\frac{\pi}{6!\cdot 2}x^6+\frac{1}{2\cdot 7!}x^7 \qquad (a=7)\\
&=\frac{2^7\pi^8}{8!}B_4 x-\frac{2^5\pi^6}{3!\cdot 6!}B_3 x^3+\frac{2^3\pi^4}{5!\cdot 4!}B_2 x^5-\frac{2\pi^2}{7!\cdot 2!}B_1 x^7+\frac{\pi}{8!\cdot 2}x^8-\frac{1}{2\cdot 9!}x^9 \qquad (a=9)\\
&\qquad\vdots\\
&=(-1)^c\left(\frac{(2c)!}{x^{2c+1}}-\frac{B_{c+1}}{1!\cdot(2c+2)}x-\frac{B_{c+2}}{3!\cdot(2c+4)}x^3-\frac{B_{c+3}}{5!\cdot(2c+6)}x^5-\cdots\right) \quad (a=-2c,\ c=1,2,3,\cdots)\\
&\left(=\frac{1}{2}\cot\frac{x}{2} \quad (a=0)\right)\\
&\qquad\vdots
\end{aligned}
\tag{19}
$$

これらは $a$ が正の偶数の場合には係数が発散する。$a=-1,-3,-5,\cdots$ の場合には $=0$ となる。

$$
\begin{aligned}
&\sin x - \frac{1}{2^a}\sin 2x + \frac{1}{3^a}\sin 3x - \frac{1}{4^a}\sin 4x + \cdots \\
&= \frac{1}{2}\tan\frac{x}{2} \quad (a=0) \\
&= \ln 2\cdot x + (1-2^2)\frac{B_1}{2\cdot 3!}x^3 + (1-2^4)\frac{B_2}{4\cdot 5!}x^5 + (1-2^6)\frac{B_3}{6\cdot 7!}x^7 + \cdots \quad (a=2) \\
&= \frac{3}{4}\zeta(3)x - \frac{\ln 2}{6}x^3 - (1-2^2)\frac{B_1}{5!\cdot 2}x^5 - (1-2^4)\frac{B_2}{7!\cdot 4}x^7 - (1-2^6)\frac{B_3}{9!\cdot 6}x^9 - \cdots \quad (a=4) \\
&= \frac{15}{16}\zeta(5)x - \frac{1}{8}\zeta(3)x^3 + \frac{\ln 2}{5!}x^5 + (1-2^2)\frac{B_1}{7!\cdot 2}x^7 + (1-2^4)\frac{B_2}{9!\cdot 4}x^9 + \cdots \quad (a=6) \\
&= \frac{2^6-1}{2^6}\zeta(7)x - \frac{2^4-1}{3!\cdot 2^4}\zeta(5)x^3 + \frac{2^2-1}{5!\cdot 2^2}\zeta(3)x^5 - \frac{\ln 2}{7!}x^7 - (1-2^2)\frac{B_1}{9!\cdot 2}x^9 - (1-2^4)\frac{B_2}{11!\cdot 4}x^{11} - \cdots \quad (a=8) \\
&= \frac{2^8-1}{2^8}\zeta(9)x - \frac{2^6-1}{3!\cdot 2^6}\zeta(7)x^3 + \frac{2^4-1}{5!\cdot 2^4}\zeta(5)x^5 - \frac{2^2-1}{7!\cdot 2^2}\zeta(3)x^7 + \frac{\ln 2}{9!}x^9 + (1-2^2)\frac{B_1}{11!\cdot 2}x^{11} \\
&\qquad + (1-2^4)\frac{B_1}{13!\cdot 4}x^{13} - \cdots \quad (a=10) \\
&\qquad \vdots \\
&= \frac{1}{2}x \quad (a=1) \\
&= \frac{(2-1)\pi^2}{1!\cdot 2!}B_1 x - \frac{1}{2\cdot 3!}x^3 \quad (a=3) \\
&= \frac{(2^3-1)\pi^4}{1!\cdot 4!}B_2 x - \frac{(2-1)\pi^2}{3!\cdot 2!}B_1 x^3 + \frac{1}{2\cdot 5!}x^5 \quad (a=5) \\
&= \frac{(2^5-1)\pi^6}{1!\cdot 6!}B_3 x - \frac{(2^3-1)\pi^4}{3!\cdot 4!}B_2 x^3 + \frac{(2-1)\pi^2}{5!\cdot 2!}B_1 x^5 - \frac{1}{2\cdot 7!}x^7 \quad (a=7) \\
&\qquad \vdots \\
&= (-1)^c\left(\frac{(4^{c+1}-1)B_{c+1}}{1!\cdot(2c+2)}x + \frac{(4^{c+2}-1)B_{c+2}}{3!\cdot(2c+4)}x^3 + \frac{(4^{c+3}-1)B_{c+3}}{5!\cdot(2c+6)}x^5 + \cdots\right) \quad (a=-2c,\ c=0,1,2,3,\cdots) \\
&\qquad \vdots
\end{aligned} \tag{20}
$$

これらは $a=-1,-3,-5,\cdots$ で $=0$ となる。

$$
\begin{aligned}
&\sin x + \frac{1}{3^a}\sin 3x + \frac{1}{5^a}\sin 5x + \frac{1}{7^a}\sin 7x + \frac{1}{9^a}\sin 9x \cdots \\
&= \frac{\pi}{4} \quad (a=1) \\
&= -\frac{\pi}{4\cdot 2!}x^2 + \frac{\pi^2}{8}x \quad (a=3) \\
&= \frac{\pi}{4\cdot 4!}X^4 - \frac{\pi^2}{3!\cdot 8}x^3 + \frac{(2^4-1)\pi^4}{4!\cdot 2}B_2 x \quad (a=5) \\
&= -\frac{\pi}{6!\cdot 4}x^6 + \frac{\pi^2}{5!\cdot 8}x^5 - \frac{(2^4-1)\pi^4}{3!\cdot 4\cdot 2}B_2 x^3 + \frac{(2^6-1)\pi^6}{6!\cdot 2}B_3 x \quad (a=7) \\
&\qquad \vdots \\
&= (-1)^c\left(\frac{1}{2}\frac{(2c)!}{x^{2c+1}} + \frac{(2^{2c+1}-1)B_{c+1}}{1!\cdot(2c+2)}x + \frac{(2^{2c+3}-1)B_{c+2}}{3!\cdot(2c+4)}x^3 + \frac{(2^{2c+5}-1)B_{c+3}}{5!\cdot(2c+6)}x^5 + \cdots\right) \quad (a=-2c,\ c=0,1,2,3,\cdots) \\
&\left(= \frac{1}{2\sin x} \quad (a=0)\right) \\
&\qquad \vdots
\end{aligned} \tag{21}
$$

これらは $a=-1,-3,-5,-7,\cdots$ で $=0$ となる。$a=2,4,6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\sin x-\frac{1}{3^a}\sin 3x+\frac{1}{5^a}\sin 5x-\frac{1}{7^a}\sin 7x+\cdots\\
&=0 \quad (a=0)\\
&=\frac{\pi}{4}x \quad (a=2)\\
&=\frac{\left(\frac{\pi}{2}\right)^3}{2\cdot 2!}E_1x-\frac{\left(\frac{\pi}{2}\right)}{2\cdot 3!\cdot 0!}E_0x^3 \quad (a=4)\\
&=\frac{\left(\frac{\pi}{2}\right)^5}{2!\cdot 1!\cdot 4!}E_2x-\frac{\left(\frac{\pi}{2}\right)^3}{2\cdot 3!\cdot 2!}E_1x^3+\frac{\left(\frac{\pi}{2}\right)}{2\cdot 5!\cdot 0!}E_0x^5 \quad (a=6)\\
&=\frac{\left(\frac{\pi}{2}\right)^7}{2\cdot 1!\cdot 6!}E_3x-\frac{\left(\frac{\pi}{2}\right)^5}{2\cdot 3!\cdot 4!}E_2x^3+\frac{\left(\frac{\pi}{2}\right)^3}{2\cdot 5!\cdot 2!}E_1x^5-\frac{\left(\frac{\pi}{2}\right)}{2\cdot 7!\cdot 0!}E_0x^7 \quad (a=8)\\
&\qquad\vdots\\
&=\mu(0)x+\frac{E_1}{2\cdot 3!}x^3+\frac{E_2}{2\cdot 5!}x^5+\frac{E_3}{2\cdot 7!}x^7+\cdots \quad (a=1)\\
&=\mu(2)x-\frac{\mu(0)}{3!}x^3-\frac{E_1}{2\cdot 5!}x^5-\frac{E_2}{2\cdot 7!}x^7-\cdots \quad (a=3)\\
&=\mu(4)x-\frac{\mu(2)}{3!}x^3+\frac{\mu(0)}{5!}x^5+\frac{E_1}{2\cdot 7!}x^7+\cdots \quad (a=5)\\
&\qquad\vdots\\
&=(-1)^{c+1}\left(\frac{E_{c+1}}{1!\cdot 2}x+\frac{E_{c+2}}{3!\cdot 2}x^3+\frac{E_{c+3}}{5!\cdot 2}x^5+\cdots\right) \quad (a=-(2c+1),\ c=-1,0,1,2,3,\cdots)
\end{aligned}
\tag{22}
$$

これは $a=0,-2,-4,-6,\cdots$ で $=0$ となる。

$$
\begin{aligned}
&\cos x+\frac{1}{2^a}\cos 2x+\frac{1}{3^a}\cos 3x+\frac{1}{4^a}\cos 4x+\frac{1}{5^a}\cos 5x+\cdots\\
&=\frac{2\pi^2}{0!\cdot 2!}B_1-\frac{\pi}{1!\cdot 2}x+\frac{1}{2\cdot 2!}x^2 \quad (a=2)\\
&=\frac{2^3\pi^4}{0!\cdot 4!}B_2-\frac{2\pi^2}{2!\cdot 2!}B_1x^2+\frac{\pi}{3!\cdot 2}x^3-\frac{1}{2\cdot 4!}x^4 \quad (a=4)\\
&=\frac{2^5\pi^6}{01\cdot 6!}B_3-\frac{2^3\pi^4}{2!\cdot 4!}B_2x^2+\frac{2\pi^2}{4!\cdot 2!}B_1x^4-\frac{\pi}{2\cdot 5!}x^5+\frac{1}{2\cdot 6!}x^6 \quad (a=6)\\
&=\frac{2^7\pi^8}{0!\cdot 8!}B_4-\frac{2^5\pi^6}{2!\cdot 6!}B_3x^2+\frac{2^3\pi^4}{4!\cdot 4!}B_2x^4-\frac{2\pi^2}{6!\cdot 2!}B_1x^6+\frac{\pi}{2\cdot 7!}x^7-\frac{1}{2\cdot 8!}x^8 \quad (a=8)\\
&\qquad\vdots\\
&=-\frac{1}{x^2}-\frac{B_1}{2\cdot 0!}-\frac{B_2}{4\cdot 2!}x^2-\frac{B_3}{6\cdot 4!}x^4-\frac{B_4}{8\cdot 6!}x^6-\cdots \quad (a=-1)\\
&=\frac{3!}{x^4}+\frac{B_2}{4\cdot 0!}+\frac{B_3}{6\cdot 2!}x^2+\frac{B_4}{8\cdot 4!}x^4+\frac{B_5}{10\cdot 6!}x^6+\cdots \quad (a=-3)\\
&=-\frac{5!}{x^6}-\frac{B_3}{6\cdot 0!}-\frac{B_4}{8\cdot 2!}x^2-\frac{B_5}{10\cdot 4!}x^4-\frac{B_6}{12\cdot 6!}x^6-\cdots \quad (a=-5)\\
&\qquad\vdots
\end{aligned}
\tag{23}
$$

これは $a=-2,-4,-6,\cdots$ で $=0$、$a=0$ で $=-\frac{1}{2}$ となる。$a=1,3,5,7,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\cos x + \frac{1}{3^a}\cos 3x + \frac{1}{5^a}\cos 5x + \frac{1}{7^a}\cos 7x + \cdots \\
&= \frac{(2^2-1)\pi^2}{2\cdot 0!\cdot 2!}B_1 - \frac{\pi}{4\cdot 1!}x \quad (a=2) \\
&= \frac{(2^4-1)\pi^4}{2\cdot 0!\cdot 4!}B_2 - \frac{(2^2-1)\pi^2}{2\cdot 2!\cdot 2!}B_1x^2 + \frac{\pi}{4\cdot 3!}x^3 \quad (a=4) \\
&= \frac{(2^6-1)\pi^6}{2\cdot 0!\cdot 6!}B_3 - \frac{(2^4-1)\pi^4}{2\cdot 2!\cdot 4!}B_2x^2 + \frac{(2^2-1)\pi^2}{2\cdot 4!\cdot 2!}B_1x^4 - \frac{\pi}{4\cdot 5!} \quad (a=6) \\
&\qquad \vdots \\
&= -\frac{1}{2x^2} + \frac{B_1}{2\cdot 0!} + \frac{(2^3-1)}{4\cdot 2!}B_2x^2 + \frac{(2^5-1)}{6\cdot 4!}B_3x^4 + \frac{(2^7-1)}{8\cdot 6!}B_4x^6 + \cdots \quad (a=-1) \\
&= \frac{3!}{2x^4} - \frac{(2^3-1)}{4\cdot 0!}B_2 - \frac{(2^5-1)}{6\cdot 2!}B_3x^2 - \frac{(2^7-1)}{8\cdot 4!}B_4x^4 - \frac{(2^9-1)}{10\cdot 6!}B_5x^6 - \cdots \quad (a=-3) \\
&= -\frac{5!}{2x^6} + \frac{(2^5-1)}{6\cdot 0!}B_3 + \frac{(2^7-1)}{8\cdot 2!}B_4x^2 + \frac{(2^9-1)}{10\cdot 4!}B_5x^4 + \frac{(2^{11}-1)}{12\cdot 6!}B_6x^6 + \cdots \quad (a=-5) \\
&\qquad \vdots
\end{aligned}
\tag{24}
$$

これは $a=0,-2,-4,-6,\cdots$ で $=0$ となる。$a=1,3,5,7,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\cos x - \frac{1}{2^a}\cos 2x + \frac{1}{3^a}\cos 3x - \frac{1}{4^a}\cos 4x + \frac{1}{5^a}\cos 5x - \cdots \\
&= \frac{\pi^2}{0!\cdot 2!}B_1 - \frac{1}{2\cdot 2!}x^2 \quad (a=2) \\
&= \frac{(2^3-1)\pi^4}{0!\cdot 4!}B_2 - \frac{\pi^2}{2!\cdot 2!}B_1x^2 + \frac{1}{2\cdot 4!}x^4 \quad (a=4) \\
&= \frac{(2^5-1)\pi^6}{0!\cdot 6!}B_3 - \frac{(2^3-1)\pi^4}{2!\cdot 4!}B_2x^2 + \frac{\pi^2}{4!\cdot 2!}B_1x^4 - \frac{1}{2\cdot 6!}x^6 \quad (a=6) \\
&\qquad \vdots \\
&= \ln 2 - \frac{2^2-1}{2\cdot 2!}B_1x^2 - \frac{2^4-1}{4\cdot 4!}B_2x^4 - \frac{2^6-1}{6\cdot 6!}B_3x^6 - \cdots \quad (a=1) \\
&= \frac{2^2-1}{2^2}\zeta(3) - \frac{\ln 2}{2!}x^2 + \frac{2^2-1}{2\cdot 4!}B_1x^4 + \frac{2^4-1}{4\cdot 6!}B_2x^6 + \cdots \quad (a=3) \\
&= \frac{2^4-1}{2^4}\zeta(5) - \frac{2^2-1}{2^2\cdot 2!}\zeta(3)x^2 + \frac{\ln 2}{4!}x^4 - \frac{2^2-1}{2\cdot 6!}B_1x^6 - \cdots \quad (a=5) \\
&= \frac{2^6-1}{2^6}\zeta(7) - \frac{2^4-1}{2^4\cdot 2!}\zeta(5)x^2 + \frac{2^2-1}{2^2\cdot 4!}\zeta(3)x^4 - \frac{\ln 2}{6!}x^6 + \cdots \quad (a=7) \\
&\qquad \vdots \\
&= \frac{2^2-1}{2\cdot 0!}B_1 + \frac{2^4-1}{4\cdot 2!}B_2x^2 + \frac{2^6-1}{6\cdot 4!}B_3x^4 + \frac{2^8-1}{8\cdot 6!}B_4x^6 + \cdots \quad (a=-1) \\
&= -\frac{2^4-1}{4\cdot 0!}B_2 - \frac{2^6-1}{6\cdot 2!}B_3x^2 - \frac{2^8-1}{8\cdot 4!}B_4x^4 - \frac{2^{10}-1}{10\cdot 6!}B_5x^6 - \cdots \quad (a=-3) \\
&= \frac{2^6-1}{6\cdot 0!}B_3 + \frac{2^8-1}{8\cdot 2!}B_4x^2 + \frac{2^{10}-1}{10\cdot 4!}x^4 + \frac{2^{12}-1}{12\cdot 6!}B_6x^6 + \cdots \quad (a=-5) \\
&\qquad \vdots
\end{aligned}
\tag{25}
$$

これは $a=-2,-4,-6,\cdots$ で $=0$ となり、$a=0$ で $=\frac{1}{2}$ となる。

$$
\begin{aligned}
&\cos x - \frac{1}{3^a}\cos 3x + \frac{1}{5^a}\cos 5x - \frac{1}{7^a}\cos 7x + \frac{1}{9^a}\cos 9x - \cdots \\
&= \frac{\pi}{4} \quad (a=1) \\
&= \frac{\pi^3}{2^4\cdot 2!}E_1 - \frac{\pi}{4\cdot 2!}x^2 \quad (a=3) \\
&= \frac{\pi^5}{2^6\cdot 4!}E_2 - \frac{\pi^3}{2^4\cdot 2!\cdot 2!}E_1x^2 + \frac{\pi}{4\cdot 4!}x^4 \quad (a=5) \\
&= \frac{\pi^7}{2^8\cdot 6!}E_3 - \frac{\pi^5}{2^6\cdot 4!\cdot 2!}E_2x^2 + \frac{\pi^3}{2^4\cdot 2!\cdot 4!}E_1x^4 - \frac{\pi}{4\cdot 6!}x^6 \quad (a=7) \\
&\quad\vdots \\
&= \mu(2) - \frac{1}{2\cdot 2!}x^2 - \frac{1}{2\cdot 2!}x^2 - \frac{E_1}{2\cdot 4!}x^4 - \frac{E_2}{2\cdot 6!}x^6 - \frac{E_3}{2\cdot 8!}x^8 - \cdots \quad (a=2) \\
&= \mu(4) - \frac{1}{2!}\mu(2)x^2 + \frac{1}{2\cdot 4!}x^4 + \frac{E_1}{2\cdot 6!}x^6 + \frac{E_2}{2\cdot 8!}x^8 + \cdots \quad (a=4) \\
&= \mu(6) - \frac{1}{2!}\mu(4)x^2 + \frac{1}{4!}\mu(2)x^4 - \frac{1}{2\cdot 6!}x^6 - \frac{E_1}{2\cdot 8!}x^8 - \cdots \quad (a=6) \\
&\quad\vdots \\
&= \frac{1}{2} + \frac{E_1}{2\cdot 2!}x^2 + \frac{E_2}{2\cdot 4!}x^4 + \frac{E_3}{2\cdot 6!}x^6 + \frac{E_4}{2\cdot 8!}x^8 + \cdots \quad (a=0) \\
&= -\frac{E_1}{2} - \frac{E_2}{2\cdot 2!}x^2 - \frac{E_3}{2\cdot 4!}x^4 - \frac{E_4}{2\cdot 6!}x^6 - \frac{E_5}{2\cdot 8!}x^8 - \cdots \quad (a=-2) \\
&= \frac{E_1}{2} + \frac{E_3}{2\cdot 2!}x^2 + \frac{E_4}{2\cdot 4!}x^4 + \frac{E_5}{2\cdot 6!}x^6 + \frac{E_6}{2\cdot 8!}x^8 + \cdots \quad (a=-4) \\
&\quad\vdots
\end{aligned}
\tag{26}
$$

これは $a=-1,-3,-5,\cdots$ で $=0$ となる。

## 5 ディリクレ級数の１例

上で求めた公式をさらに利用して

$$
\begin{aligned}
\frac{\sin x}{1-u^2} + \frac{2\sin 2x}{2^2-u^2} + \frac{3\sin 3x}{3^2-u^2} + \frac{4\sin 4x}{4^2-u^2} + \frac{5\sin 5x}{5^2-u^2} + \cdots &= \frac{\pi}{2}\cos xu - \frac{\pi}{2}\cot \pi u \cdot \sin xu \\
&= \frac{\pi}{2}\frac{\sin\{(\pi-x)u\}}{\sin \pi u}
\end{aligned}
\tag{27}
$$

これは

$$
\sum_{n=1}\frac{n\sin nx}{n^2-u^2} = \sum_{m=0}^{\infty}u^{2m}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m+1}} \tag{28}
$$

を (19) を用いて $u$ と $x$ の冪級数で表し、それらから係数を比較し右辺を求める。同様の計算で

$$
\frac{\sin x}{1-u^2} - \frac{2\sin 2x}{2^2-u^2} + \frac{3\sin 3x}{3^2-u^2} - \frac{4\sin 4x}{4^2-u^2} + \frac{5\sin 5x}{5^2-u^2} - \cdots = \frac{\pi}{2}\frac{\sin xu}{\sin \pi u} \tag{29}
$$

$$
\begin{aligned}
\frac{\sin x}{1-u^2} + \frac{3\sin 3x}{3^2-u^2} + \frac{5\sin 5x}{5^2-u^2} + \frac{7\sin 7x}{7^2-u^2} + \frac{9\sin 9x}{9^2-u^2} + \cdots &= \frac{\pi}{4}\cos xu + \frac{\pi}{4}\tan\frac{\pi}{2}u \cdot \sin ux \\
&= \frac{\pi}{4}\frac{\cos\{(x-\frac{\pi}{2})u\}}{\cos\frac{\pi}{2}u}
\end{aligned}
\tag{30}
$$

$$\frac{\sin x}{1-u^2}-\frac{\sin 3x}{3^2-u^2}+\frac{\sin 5x}{5^2-u^2}-\frac{\sin 7x}{7^2-u^2}+\frac{\sin 9x}{9^2-u^2}-\cdots=\frac{\pi\sin xu}{4u\cos\frac{\pi}{2}u} \tag{31}$$

$$\begin{aligned}\frac{\cos x}{1-u^2}+\frac{2^2\cos 2x}{2^2-u^2}+\frac{3^2\cos 3x}{3^2-u^2}+\frac{4^2\cos 4x}{4^2-u^2}+\frac{5^2\cos 5x}{5^2-u^2}+\cdots &= -\frac{\pi u}{2}\Big(\sin xu+\cot\pi u\cdot\cos xu\Big)\\ &= -\frac{\pi u}{2}\frac{\cos\{(x-\pi)u\}}{\sin\pi u}\end{aligned} \tag{32}$$

$$\frac{\cos x}{1-u^2}-\frac{2^2\cos 2x}{2^2-u^2}+\frac{3^2\cos 3x}{3^2-u^2}-\frac{4^2\cos 4x}{4^2-u^2}+\frac{5^2\cos 5x}{5^2-u^2}-\cdots=\frac{\pi u}{2}\frac{\cos xu}{\sin\pi u} \tag{33}$$

$$\begin{aligned}\frac{\cos x}{1-u^2}+\frac{\cos 3x}{3^2-u^2}+\frac{\cos 5x}{5^2-u^2}+\frac{\cos 7x}{7^2-u^2}+\frac{\cos 9x}{9^2-u^2}+\cdots &= \frac{\pi}{4u}\Big(\tan\frac{\pi u}{2}\cdot\cos xu-\sin xu\Big)\\ &= \frac{\pi}{4u}\frac{\sin\{(\frac{\pi}{2}-x)u\}}{\cos\frac{\pi}{2}u}\end{aligned} \tag{34}$$

$$\frac{\cos x}{1-u^2}-\frac{3\cos 3x}{3^2-u^2}+\frac{5\cos 5x}{5^2-u^2}-\frac{7\cos 7x}{7^2-u^2}+\frac{9\cos 9x}{9^2-u^2}-\cdots=\frac{\pi}{4}\frac{\cos xu}{\cos\frac{\pi}{2}u} \tag{35}$$

次にゼータ関数の積分表示を応用して級数の積分表示を求める。

$$\int_0^\infty\frac{x^{s-1}}{e^{bx}(e^{ax}-1)}dx=\Gamma(s)\Big(\frac{1}{(b+a)^s}+\frac{1}{(b+2a)^s}+\frac{1}{(b+3a)^s}+\cdots\Big) \tag{36}$$

$$\int_0^\infty\frac{e^{bx}x^{s-1}}{e^{ax}-e^{-ax}}dx=\Gamma(s)\Big(\frac{1}{(a-b)^s}+\frac{1}{(3a-b)^s}+\frac{1}{(5a-b)^s}+\cdots\Big) \tag{37}$$

$$\begin{aligned}\int_0^\infty\frac{\cosh bx}{\sinh ax}x^{s-1}dx &= \int_0^\infty\frac{e^{bx}+e^{-bx}}{e^{ax}-e^{-ax}}x^{s-1}dx\\ &= \Gamma(s)\Big(\frac{1}{(a-b)^s}+\frac{1}{(3a-b)^s}+\frac{1}{(5a-b)^s}+\cdots\\ &\qquad +\frac{1}{(a+b)^s}+\frac{1}{(3a+b)^s}+\frac{1}{(5a+b)^s}+\cdots\Big)\end{aligned} \tag{38}$$

$$\begin{aligned}\int_0^\infty\frac{\sinh bx}{\sinh ax}x^{s-1}dx &= \Gamma(s)\Big(\frac{1}{(a-b)^s}+\frac{1}{(3a-b)^s}+\frac{1}{(5a-b)^s}+\cdots\\ &\qquad -\frac{1}{(a+b)^2}-\frac{1}{(3a+b)^s}-\frac{1}{(5a+b)^s}-\cdots\Big)\end{aligned} \tag{39}$$

などを得る。

次に上で求めた公式からゼータ関数の形の級数の積分表示を求める。例えば

$$\begin{aligned}\int_0^\infty u^{s-1}\frac{\sinh\{(\pi-x)u\}}{\sinh\pi u}du &= \frac{2}{\pi}\int_0^\infty u^{s-1}\sum_{n=1}\frac{n\sin nx}{n^2+u^2}du\\ &= \frac{2}{\pi}\sum_{n=1}n\sin nx\int_0^\infty\frac{u^{s-1}}{n^2+u^2}du\\ &= \frac{1}{\sin\frac{\pi}{2}s}\sum_{n=1}^\infty\frac{\sin nx}{n^{1-s}}\end{aligned} \tag{40}$$

として求められる。同様の計算をすることで似た形の他のゼータ関数の形の級数の積分表示を求められる。まとめると

$$\phi_1(s) = \frac{\sin x}{1^s} + \frac{\sin 2x}{2^s} + \frac{\sin 3x}{3^s} + \frac{\sin 4x}{4^s} + \cdots \tag{41}$$

$$\phi_2(s) = \frac{\sin x}{1^s} - \frac{\sin 2x}{2^s} + \frac{\sin 3x}{3^s} - \frac{\sin 4x}{4^s} + \cdots \tag{42}$$

$$\phi_3(s) = \frac{\sin x}{1^s} + \frac{\sin 3x}{3^s} + \frac{\sin 5x}{5^s} + \frac{\sin 7x}{7^s} + \cdots \tag{43}$$

$$\phi_4(s) = \frac{\sin x}{1^s} - \frac{\sin 3x}{3^s} + \frac{\sin 5x}{5^s} - \frac{\sin 7x}{7^s} + \cdots \tag{44}$$

$and$

$$\psi_1(s) = \frac{\cos x}{1^s} + \frac{\cos 2x}{2^s} + \frac{\cos 3x}{3^s} + \frac{\cos 4x}{4^s} + \cdots \tag{45}$$

$$\psi_2(s) = \frac{\cos x}{1^s} - \frac{\cos 2x}{2^s} + \frac{\cos 3x}{3^s} - \frac{\cos 4x}{4^s} + \cdots \tag{46}$$

$$\psi_3(s) = \frac{\cos x}{1^s} + \frac{\cos 3x}{3^s} + \frac{\cos 5x}{5^s} + \frac{\cos 7x}{7^s} + \cdots \tag{47}$$

$$\psi_4(s) = \frac{\cos x}{1^s} - \frac{\cos 3x}{3^s} + \frac{\cos 5x}{5^s} - \frac{\cos 7x}{7^s} + \cdots \tag{48}$$

$$\tag{49}$$

と置いた時

$$\phi_1(1-s) = \sin\frac{\pi}{2}s \int_0^\infty u^{s-1}\frac{\sinh\{(\pi-x)u\}}{\sinh \pi u}du \tag{50}$$

$$\phi_2(1-s) = \sin\frac{\pi}{2}s \int_0^\infty u^{s-1}\frac{\sinh xu}{\sinh \pi u}du \tag{51}$$

$$\phi_3(1-s) = \frac{\sin\frac{\pi}{2}s}{2} \int_0^\infty u^{s-1}\frac{\cosh\{(x-\frac{\pi}{2})u\}}{\cosh \frac{\pi}{2}u}du \tag{52}$$

$$\phi_4(1-s) = \frac{\cos\frac{\pi}{2}s}{2} \int_0^\infty u^{s-1}\frac{\sinh xu}{\cosh \frac{\pi}{2}u}du \tag{53}$$

$and$

$$\psi_1(1-s) = \cos\frac{\pi}{2}s \int_0^\infty u^{s-1}\frac{\cosh\{(x-\pi)u\}}{\sinh \pi u}du \tag{54}$$

$$\psi_2(1-s) = -\cos\frac{\pi}{2}s \int_0^\infty u^{s-1}\frac{\cosh xu}{\sinh \pi u}du \tag{55}$$

$$\psi_3(1-s) = \frac{\cos\frac{\pi}{2}s}{2} \int_0^\infty u^{s-1}\frac{\sinh\{(\frac{\pi}{2}-x)u\}}{\cosh \frac{\pi}{2}u}du \tag{56}$$

$$\psi_4(1-s) = \frac{\sin\frac{\pi}{2}s}{2} \int_0^\infty u^{s-1}\frac{\cosh xu}{\cosh \frac{\pi}{2}u}du \tag{57}$$

次に

$$\begin{aligned}
\psi_1(s) + i\phi_1(s) &= \sum_{n=1}^{\infty}\frac{e^{inx}}{n^s} \\
&= \sum_{n=1}^{\infty} e^{inx}\frac{1}{\Gamma(s)}\int_0^\infty e^{-nu}u^{s-1}du \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{ix-u}}{1-e^{ix-u}}u^{s-1}du \\
&= \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{e^{ix}-e^{-u}}{e^u - 2\cos x + e^{-u}}du
\end{aligned} \tag{58}$$

となることより、実部と虚部を比較すれば $\phi_1(s)$ と $\psi_1(s)$ の積分表示も得られる。他も同様である。まとめて書くと

$$\phi_1(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\sin x}{e^u - 2\cos x + e^{-u}}du \tag{59}$$

$$\phi_2(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\sin x}{e^u + 2\cos x + e^{-u}}du \tag{60}$$

$$\phi_3(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\sin x(e^u + e^{-u})}{e^{2u} - 2\cos 2x + e^{-2u}}du \tag{61}$$

$$\phi_4(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\sin x(e^u - e^{-u})}{e^{2u} + 2\cos 2x + e^{-2u}}du \tag{62}$$

*and*

$$\psi_1(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\cos x - e^{-u}}{e^u - 2\cos x + e^{-u}}du \tag{63}$$

$$\psi_2(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\cos x + e^{-u}}{e^u + 2\cos x + e^{-u}}du \tag{64}$$

$$\psi_3(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\cos x(e^u - e^{-u})}{e^{2u} - 2\cos 2x + e^{-2u}}du \tag{65}$$

$$\psi_4(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\frac{\cos x(e^u + e^{-u})}{e^{2u} + 2\cos 2x + e^{-2u}}du \tag{66}$$

となる。ここで (50) から (57) 式までに対して、(36) から (39) などを用いると、

$$\phi_1(1-s) = \Gamma(s)\sin\frac{\pi}{2}s\left(\frac{1}{x^s} + \frac{1}{(2\pi+x)^s} - \frac{1}{(2\pi-x)^s} + \frac{1}{(4\pi+x)^s} - \frac{1}{(4\pi-x)^s} + \cdots\right) \tag{67}$$

$$\phi_2(1-s) = \Gamma(s)\sin\frac{\pi}{2}s\left(\frac{1}{(\pi-x)^s} - \frac{1}{(\pi+x)^s} + \frac{1}{(3\pi-x)^s} - \frac{1}{(3\pi+x)^s} + \frac{1}{(5\pi-x)^s} - \frac{1}{(5\pi+x)^s} + \cdots\right) \tag{68}$$

$$\phi_3(1-s) = \frac{1}{2}\Gamma(s)\sin\frac{\pi}{2}s\left(\frac{1}{x^s} - \frac{1}{(\pi+x)^s} + \frac{1}{(\pi-x)^s} + \frac{1}{(2\pi+x)^s} - \frac{1}{(2\pi-x)^s} + \cdots\right) \tag{69}$$

$$\phi_4(1-s) = \frac{1}{2}\Gamma(s)\cos\frac{\pi}{2}s\left(\frac{1}{\left(\frac{\pi}{2}-x\right)^s} - \frac{1}{\left(\frac{\pi}{2}+x\right)} - \frac{1}{\left(\frac{3}{2}\pi-x\right)^s} + \frac{1}{\left(\frac{3}{2}\pi+x\right)^s} + \frac{1}{\left(\frac{5}{2}\pi-x\right)^s} - \frac{1}{\left(\frac{5}{2}\pi+x\right)^s} + \cdots\right) \tag{70}$$

*and*

$$\psi_1(1-s) = \Gamma(s)\cos\frac{\pi}{2}s\left(\frac{1}{x^s} + \frac{1}{(2\pi-x)^s} + \frac{1}{(2\pi+x)^s} + \frac{1}{(4\pi-x)^s} + \frac{1}{(4\pi+x)^s} + \cdots\right) \tag{71}$$

$$\psi_2(1-s) = -\Gamma(s)\cos\frac{\pi}{2}s\left(\frac{1}{(\pi-x)^s} + \frac{1}{(\pi+x)^s} + \frac{1}{(3\pi-x)^s} + \frac{1}{(3\pi+x)^s} + \frac{1}{(5\pi-x)^s} + \frac{1}{(5\pi+x)^s} + \cdots\right) \tag{72}$$

$$\psi_3(1-s) = \frac{1}{2}\Gamma(s)\cos\frac{\pi}{2}s\left(\frac{1}{x^s} - \frac{1}{(\pi+x)^s} - \frac{1}{(\pi-x)^s} + \frac{1}{(2\pi+x)^s} + \frac{1}{(2\pi-x)^s} + \cdots\right) \tag{73}$$

$$\psi_4(1-s) = \frac{1}{2}\Gamma(s)\sin\frac{\pi}{2}s\left(\frac{1}{\left(\frac{\pi}{2}+x\right)^s} + \frac{1}{\left(\frac{\pi}{2}-x\right)^s} - \frac{1}{\left(\frac{3}{2}\pi+x\right)^s} - \frac{1}{\left(\frac{3}{2}\pi-x\right)^s} + \frac{1}{\left(\frac{5}{2}\pi+x\right)^s} + \frac{1}{\left(\frac{5}{2}\pi-x\right)^s} + \cdots\right) \tag{74}$$

を得る。

## 6　いろいろな級数を多項式やベキ級数で表現する

ここまでのことを踏まえていろいろな級数を多項式で表現していく。

$$
\begin{aligned}
&\frac{1}{e^x-1}+\frac{2^a}{e^{2x}-1}+\frac{3^a}{e^{3x}-1}+\frac{4^a}{e^{4x}-1}+\frac{5^a}{e^{5x}-1}+\dots\\
&=\frac{(2\pi)^2}{2\cdot 2}B_1\frac{1}{x^2}-\frac{1}{2x}+\frac{1}{2\cdot 2}B_1 \quad (a=1)\\
&=\frac{(2\pi)^4}{2\cdot 4}B_2\frac{1}{x^4}-\frac{1}{2\cdot 4}B_2 \quad (a=3)\\
&=\frac{(2\pi)^6}{2\cdot 6}B_3\frac{1}{x^6}+\frac{1}{2\cdot 6}B_3 \quad (a=5)\\
&=\frac{(2\pi)^8}{2\cdot 8}B_4\frac{1}{x^8}-\frac{1}{2\cdot 8}B_4 \quad (a=7)\\
&\vdots\\
&=2!\cdot\zeta(3)\frac{1}{x^3}-\frac{B_1}{2x}+\frac{B_1B_2}{1!\cdot 2\cdot 4}x+\frac{B_2B_3}{3!\cdot 4\cdot 6}x^3+\frac{B_3B_4}{5!\cdot 6\cdot 8}x^5+\dots \quad (a=2)\\
&=4!\cdot\zeta(5)\frac{1}{x^5}+\frac{B_2}{4x}-\frac{B_1B_3}{1!\cdot 2\cdot 6}x-\frac{B_2B_4}{3!\cdot 4\cdot 8}x^3-\frac{B_3B_5}{5!\cdot 6\cdot 10}x^5-\dots \quad (a=4)\\
&=6!\cdot\zeta(7)\frac{1}{x^7}-\frac{B_3}{6x}+\frac{B_1B_4}{1!\cdot 2\cdot 8}x+\frac{B_2B_5}{3!\cdot 4\cdot 10}x^3+\frac{B_3B_6}{5!\cdot 6\cdot 12}x^5+\dots \quad (a=6)\\
&\vdots\\
&=\frac{(2\pi)^4}{2\cdot 4!}B_2\frac{1}{x}-\frac{1}{2}\zeta(3)+\frac{(2\pi)^2B_1^2}{2\cdot 2\cdot 2!\cdot 1!}x+\frac{B_2}{2\cdot 4\cdot 3!}x^3-\frac{\zeta(3)}{2\cdot(2\pi)^2}x^2 \quad (a=-3)\\
&=\frac{(2\pi)^6}{2\cdot 6!}B_3\frac{1}{x}-\frac{1}{2}\zeta(5)+\frac{(2\pi)^4B_1B_2}{2\cdot 2\cdot 4!\cdot 1!}x-\frac{(2\pi)^2B_1B_2}{2\cdot 4\cdot 2!\cdot 3!}x^3-\frac{B_3}{2\cdot 6\cdot 5!}x^5+\frac{\zeta(5)}{2\cdot(2\pi)^4}x^4 \quad (a=-5)\\
&=\frac{(2\pi)^8}{2\cdot 8!}B_4\frac{1}{x}-\frac{1}{2}\zeta(7)+\frac{(2\pi)^6B_1B_3}{2\cdot 2\cdot 6!\cdot 1!}x-\frac{(2\pi)^4B_2^2}{2\cdot 4\cdot 3!\cdot 4!}x^3+\frac{(2\pi)^2B_1B_3}{2\cdot 6\cdot 2!\cdot 5!}x^5+\frac{B_4}{2\cdot 8\cdot 7!}x^7-\frac{\zeta(7)}{2\cdot(2\pi)^6}x^6 \quad (a=-7)\\
&\vdots
\end{aligned}
\tag{75}
$$

これらは $a=-1$ と $a=0,-2,-4,-6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^x-1}-\frac{2^a}{e^{2x}-1}+\frac{3^a}{e^{3x}-1}-\frac{4^a}{e^{4x}-1}+\frac{5^a}{e^{5x}-1}-\dots\\
&=\frac{1}{2x}-\frac{2^2-1}{2\cdot 2}B_1 \quad (a=1)\\
&=\frac{2^4-1}{2\cdot 4}B_2 \quad (a=3)\\
&=-\frac{2^6-1}{2\cdot 6}B_3 \quad (a=5)\\
&=\frac{2^8-1}{2\cdot 8}B_4 \quad (a=7)\\
&\vdots\\
&=\ln 2\cdot\frac{1}{x}-\frac{1}{4}+\frac{2^2-1}{2^2}B_1^2x+\frac{2^4-1}{3!\cdot 4^2}B_2^2x^3+\frac{2^6-1}{5!\cdot 6^2}B_3^2x^5+\dots \quad (a=0)\\
&=\frac{2^2-1}{2}B_1\frac{1}{x}-\frac{(2^4-1)B_1B_2}{2\cdot 4\cdot 1!}x-\frac{(2^6-1)B_2B_3}{4\cdot 6\cdot 3!}x^3-\frac{(2^8-1)B_3B_4}{6\cdot 8\cdot 5!}x^5-\dots \quad (a=2)\\
&=-\frac{2^4-1}{4}B_2\frac{1}{x}+\frac{(2^6-1)B_1B_3}{2\cdot 6\cdot 1!}x+\frac{(2^8-1)B_2B_4}{4\cdot 8\cdot 3!}x^3+\frac{(2^{10}-1)B_3B_5}{6\cdot 10\cdot 5!}x^5+\dots \quad (a=4)\\
&=\frac{2^6-1}{6}B_3\frac{1}{x}-\frac{(2^8-1)B_1B_4}{2\cdot 8\cdot 1!}x-\frac{(2^{10}-1)B_2B_5}{4\cdot 10\cdot 3!}x^3-\frac{(2^{12}-1)B_3B_6}{6\cdot 12\cdot 5!}x^7-\dots \quad (a=6)\\
&\vdots\\
&=\frac{(2-1)\pi^2}{2!}B_1\frac{1}{x}-\frac{\ln 2}{2}+\frac{(2-1)B_1}{2\cdot 2\cdot 1!}x \quad (a=-1)\\
&=\frac{(2^3-1)\pi^4}{4!}B_2\frac{1}{x}-\frac{2^2-1}{2^3}\zeta(3)+\frac{(2-1)\pi^2B_1B_1}{2\cdot 1!\cdot 2!}x-\frac{B_2}{2\cdot 4\cdot 3!}x^3 \quad (a=-3)\\
&=\frac{(2^5-1)\pi^6}{6!}B_3\frac{1}{x}-\frac{2^4-1}{2^5}\zeta(5)+\frac{(2^3-1)\pi^4B_1B_2}{2\cdot 1!\cdot 4!}x-\frac{\pi^2B_2B_1}{4\cdot 3!\cdot 2!}x^3+\frac{B_3}{2\cdot 6\cdot 5!}x^5 \quad (a=-5)\\
&=\frac{(2^7-1)\pi^8}{8!}B_4\frac{1}{x}-\frac{2^6-1}{2^7}\zeta(7)+\frac{(2^5-1)\pi^6B_1B_3}{2\cdot 1!\cdot 6!}x-\frac{(2^3-1)\pi^4B_2B_2}{4\cdot 3!\cdot 4!}x^3+\frac{\pi^2B_3B_1}{6\cdot 2!\cdot 5!}x^5-\frac{B_4}{2\cdot 8\cdot 7!}x^7 \quad (a=-7)\\
&\vdots\\
&=\frac{2^7-1}{2^2}\zeta(3)\frac{1}{x}-\frac{\pi^2B_1}{2\cdot 2!}+\frac{\ln 2\cdot B_1}{2\cdot 1!}x-\frac{(2^2-1)B_1B_2}{2\cdot 4\cdot 3!}x^3-\frac{(2^4-1)B_2B_3}{4\cdot 6\cdot 5!}x^5-\dots \quad (a=-2)\\
&=\frac{2^4-1}{2^4}\zeta(5)\frac{1}{x}-\frac{(2^3-1)\pi^4B_2}{2\cdot 4!}+\frac{(2^2-1)B_1}{2\cdot 2^2}\zeta(3)x-\frac{\ln 2\cdot B_2}{4\cdot 3!}x^3+\frac{(2^2-1)B_1B_3}{2\cdot 6\cdot 5!}x^5+\frac{(2^4-1)B_2B_4}{4\cdot 8\cdot 7!}x^7\dots \quad (a=-4)\\
&=\frac{2^6-1}{2^6}\zeta(7)\frac{1}{x}-\frac{(2^5-1)\pi^6B_3}{2\cdot 6!}+\frac{(2^4-1)B_1}{2\cdot 2^4}\zeta(5)x-\frac{(2^2-1)B_2}{4\cdot 2^2\cdot 3!}\zeta(3)x^3+\frac{\ln 2\cdot B_3}{6\cdot 5!}x^5-\frac{(2^2-1)B_1B_4}{2\cdot 8\cdot 7!}x^7\\
&\qquad -\frac{(2^4-1)B_2B_5}{4\cdot 10\cdot 9!}x^9-\dots \quad (a=-6)\\
&\vdots
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
&\frac{1}{e^{x}-1}+\frac{3^{a}}{e^{3x}-1}+\frac{5^{a}}{e^{5x}-1}+\frac{7^{a}}{e^{7x}-1}+\frac{9^{a}}{e^{9x}-1}+\cdots\\
&=\frac{1}{2}\frac{(2\pi)^{2}}{2\cdot 2}B_{1}\frac{1}{x^{2}}-\frac{B_{1}}{2\cdot 2}\quad (a=1)\\
&=\frac{1}{2}\frac{(2\pi)^{4}}{2\cdot 4}B_{2}\frac{1}{x^{4}}+\frac{2^{3}-1}{2}\frac{B_{2}}{4}\quad (a=3)\\
&=\frac{1}{2}\frac{(2\pi)^{6}}{2\cdot 6}B_{3}\frac{1}{x^{6}}-\frac{2^{5}-1}{2}\frac{B_{3}}{6}\quad (a=5)\\
&=\frac{1}{2}\frac{(2\pi)^{8}}{2\cdot 8}B_{4}\frac{1}{x^{8}}+\frac{2^{7}-1}{2}\frac{B_{4}}{8}\quad (a=7)\\
&\vdots\\
&=\frac{2!}{2}\zeta(3)\frac{1}{x^{3}}+\frac{B_{1}}{2}\frac{1}{x}-\frac{(2^{3}-1)B_{1}B_{2}}{2\cdot 4\cdot 1!}x-\frac{(2^{5}-1)B_{2}B_{3}}{4\cdot 6\cdot 3!}x^{3}-\cdots\quad (a=2)\\
&=\frac{4!}{2}\zeta(5)\frac{1}{x^{5}}-\frac{2^{3}-1}{4}B_{2}\frac{1}{x}+\frac{(2^{5}-1)B_{1}B_{3}}{2\cdot 6\cdot 1!}x+\frac{(2^{7}-1)B_{2}B_{4}}{4\cdot 8\cdot 3!}x^{3}+\cdots\quad (a=4)\\
&=\frac{6!}{2}\zeta(7)\frac{1}{x^{7}}+\frac{2^{5}-1}{6}B_{3}\frac{1}{x}-\frac{(2^{7}-1)B_{1}B_{4}}{2\cdot 8\cdot 1!}x-\frac{(2^{9}-1)B_{2}B_{5}}{2\cdot 10\cdot 3!}x^{3}-\cdots\quad (a=6)\\
&\vdots\\
&=\frac{(2^{4}-1)\pi^{4}}{2\cdot 4!}B_{2}\frac{1}{x}-\frac{2^{3}-1}{2^{4}}\zeta(3)+\frac{(2\pi)^{2}B_{1}B_{1}}{2\cdot 2\cdot 1!\cdot 2!}\frac{2^{2}-1}{2^{2}}x-\frac{1}{2^{4}\pi^{2}}x^{2}\quad (a=-3)\\
&=\frac{(2^{6}-1)\pi^{6}}{2\cdot 6!}B_{3}\frac{1}{x}-\frac{2^{5}-1}{2^{6}}\zeta(5)+\frac{(2\pi)^{4}B_{1}B_{2}}{2\cdot 2\cdot 1!\cdot 4!}\frac{2^{4}-1}{2^{4}}x-\frac{(2\pi)^{2}B_{2}B_{1}}{2\cdot 4\cdot 3!\cdot 2!}\frac{2^{2}-1}{2^{2}}x^{3}+\frac{1}{2^{6}\pi^{4}}x^{4}\quad (a=-5)\\
&=\frac{(2^{8}-1)\pi^{8}}{2\cdot 8!}B_{4}\frac{1}{x}-\frac{2^{7}-1}{2^{8}}\zeta(7)+\frac{(2\pi)^{6}B_{1}B_{3}}{2\cdot 2\cdot 1!\cdot 6!}\frac{2^{6}-1}{2^{6}}x-\frac{(2\pi)^{4}B_{2}B_{2}}{2\cdot 4\cdot 3!\cdot 4!}\frac{2^{4}-1}{2^{4}}x^{3}+\frac{(2\pi)^{2}B_{3}B_{1}}{2\cdot 6\cdot 5!\cdot 2!}\frac{2^{2}-1}{2^{2}}x^{5}-\frac{1}{2^{8}\pi^{6}}x^{6}\quad (a=-7)\\
&\vdots
\end{aligned}
\tag{77}
$$

これらは $a=-1$ と $a=0,-2,-4,-6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^x-1}-\frac{3^a}{e^{3x}-1}+\frac{5^a}{e^{5x}-1}-\frac{7^a}{e^{7x}-1}+\frac{9^a}{e^{9x}-1}-\cdots\\
&=\frac{E_0}{2}\frac{1}{x}-\frac{B_1E_1}{2\cdot 2\cdot 1!}x-\frac{B_2E_2}{2\cdot 4\cdot 3!}x^3-\frac{B_3E_3}{2\cdot 6\cdot 5!}x^5-\cdots \quad (a=1)\\
&=-\frac{E_1}{2}\frac{1}{x}+\frac{B_1E_2}{2\cdot 2\cdot 1!}x+\frac{B_2E_3}{2\cdot 4\cdot 3!}x^3+\frac{B_3E_4}{2\cdot 6\cdot 5!}x^5+\cdots \quad (a=3)\\
&=\frac{E_2}{2}\frac{1}{x}-\frac{B_1E_3}{2\cdot 2\cdot 1!}x-\frac{B_2E_4}{2\cdot 4\cdot 3!}x^3-\frac{B_3E_5}{2\cdot 6\cdot 5!}x^5-\cdots \quad (a=5)\\
&\vdots\\
&=\frac{\pi}{4}\frac{1}{x}-\frac{1}{4} \quad (a=0)\\
&=\frac{E_1}{4} \quad (a=2)\\
&=-\frac{E_2}{4} \quad (a=4)\\
&=\frac{E_3}{4} \quad (a=6)\\
&\vdots\\
&=\mu(2)\frac{1}{x}-\frac{\pi}{8}+\frac{B_1E_0}{2\cdot 2\cdot 1!}x+\frac{B_2E_1}{2\cdot 4\cdot 3!}x^3+\frac{B_3E_2}{2\cdot 6\cdot 5!}x^5+\cdots \quad (a=-1)\\
&=\mu(4)\frac{1}{x}-\frac{\left(\frac{\pi}{2}\right)^3}{4\cdot 2!}E_1+\frac{B_1}{2\cdot 1!}\mu(2)x-\frac{B_2E_0}{2\cdot 4\cdot 3!}x^3-\frac{B_3E_1}{2\cdot 6\cdot 5!}x^5-\cdots \quad (a=-3)\\
&=\mu(6)\frac{1}{x}-\frac{\left(\frac{\pi}{2}\right)^5}{4\cdot 4!}E_2+\frac{B_1}{2\cdot 1!}\mu(4)x-\frac{B_2}{4\cdot 3!}\mu(2)x^3+\frac{B_3E_0}{2\cdot 6\cdot 5!}x^5+\frac{B_4E_1}{2\cdot 8\cdot 7!}x^7+\cdots \quad (a=-5)\\
&\vdots\\
&=\frac{\left(\frac{\pi}{2}\right)^3}{2\cdot 2!}E_1\frac{1}{x}-\frac{1}{2}\mu(2)+\frac{\left(\frac{\pi}{2}\right)B_1E_0}{2\cdot 2\cdot 0!\cdot 1!}x \quad (a=-2)\\
&=\frac{\left(\frac{\pi}{2}\right)^5}{2\cdot 4!}E_2\frac{1}{x}-\frac{1}{2}\mu(4)+\frac{\left(\frac{\pi}{2}\right)^3B_1E_1}{2\cdot 2\cdot 2!\cdot 1!}x-\frac{\left(\frac{\pi}{2}\right)B_2E_0}{2\cdot 4\cdot 0!\cdot 3!}x^3 \quad (a=-4)\\
&=\frac{\left(\frac{\pi}{2}\right)^7}{2\cdot 6!}E_3\frac{1}{x}-\frac{1}{2}\mu(6)+\frac{\left(\frac{\pi}{2}\right)^5B_1E_2}{2\cdot 2\cdot 4!\cdot 1!}x-\frac{\left(\frac{\pi}{2}\right)^3B_2E_1}{2\cdot 4\cdot 2!\cdot 3!}x^3+\frac{\left(\frac{\pi}{2}\right)B_3E_0}{2\cdot 6\cdot 0!\cdot 5!}x^5 \quad (a=-6)\\
&\vdots
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
&\frac{1}{e^x+1}+\frac{2^a}{e^{2x}+1}+\frac{3^a}{e^{3x}+1}+\frac{4^a}{e^{4x}+1}+\frac{5^a}{e^{5x}+1}+\cdots\\
&=\frac{\pi^2}{2}B_1\frac{1}{x^2}-\frac{B_1}{2\cdot 2}\quad (a=1)\\
&=\frac{(2^3-1)\pi^4}{4}B_2\frac{1}{x^4}+\frac{B_2}{2\cdot 4}\quad (a=3)\\
&=\frac{(2^5-1)\pi^6}{6}B_3\frac{1}{x^6}-\frac{B_3}{2\cdot 6}\quad (a=5)\\
&\vdots\\
&=\ln 2\frac{1}{x}-\frac{1}{4}+\frac{(2^2-1)B_1B_1}{2\cdot 2\cdot 1!}x+\frac{(2^4-1)B_2B_2}{4\cdot 4\cdot 3!}x^3+\frac{(2^6-1)B_3B_3}{6\cdot 6\cdot 5!}x^5+\cdots\quad (a=0)\\
&=\frac{(2^2-1)2!}{2^2}\zeta(3)\frac{1}{x^3}-\frac{(2^2-1)B_1B_2}{2\cdot 4\cdot 1!}x-\frac{(2^4-1)B_2B_3}{4\cdot 6\cdot 3!}x^3-\frac{(2^6-1)B_3B_4}{6\cdot 8\cdot 5!}x^5-\cdots\quad (=2)\\
&=\frac{(2^4-1)4!}{2^4}\zeta(5)\frac{1}{x^5}+\frac{(2^2-1)B_1B_3}{2\cdot 6\cdot 1!}x+\frac{(2^4-1)B_2B_4}{4\cdot 8\cdot 3!}x^3+\frac{(2^6-1)B_3B_5}{6\cdot 10\cdot 5!}x^5+\cdots\quad (a=4)\\
&\vdots\\
&=\frac{1}{2}\zeta(3)-\frac{(2^2-1)(2\pi)^2}{2\cdot 2\cdot 1!\cdot 2!}B_1B_1x+\frac{2^3-1}{2^3\pi^2}\zeta(3)x^2-\frac{(2^4-1)B_2}{2\cdot 4\cdot 3!}x^3\quad (a=-3)\\
&=\frac{1}{2}\zeta(5)-\frac{(2^2-1)(2\pi)^4}{2\cdot 2\cdot 1!\cdot 4!}B_1B_2x+\frac{(2^4-1)(2\pi)^2}{2\cdot 4\cdot 3!\cdot 2!}B_2B_1x^3-\frac{2^5-1}{2^5\pi^4}\zeta(5)x^4+\frac{(2^6-1)B_3}{2\cdot 6\cdot 5!}x^5\quad (a=-5)\\
&=\frac{1}{2}\zeta(7)-\frac{(2^2-1)(2\pi)^6}{2\cdot 2\cdot 1!\cdot 6!}B_1B_3x+\frac{(2^4-1)(2\pi)^4}{2\cdot 4\cdot 3!\cdot 4!}B_2B_2x^3-\frac{(2^6-1)(2\pi)^2B_3B_1}{2\cdot 6\cdot 5!\cdot 2!}x^5+\frac{2^7-1}{2^7\pi^6}\zeta(7)x^6-\frac{(2^8-1)B_4}{2\cdot 8\cdot 7!}x^7\quad (a=-7)\\
&\vdots
\end{aligned}
\tag{79}
$$

これらは $a=-1$ と $a=-2,-4,-6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^x+1}-\frac{2^a}{e^{2x}+1}+\frac{3^a}{e^{3x}+1}-\frac{4^a}{e^{4x}+1}+\frac{5^a}{e^{5x}+1}-\cdots\\
&=\frac{2^2-1}{2\cdot 2}B_1 \quad (a=1)\\
&=-\frac{2^4-1}{2\cdot 4}B_2 \quad (a=3)\\
&=\frac{2^6-1}{2\cdot 6}B_3 \quad (a=5)\\
&\vdots\\
&=\frac{1}{4}-\frac{(2^2-1)^2}{2\cdot 2\cdot 1!}B_1B_1x-\frac{(2^4-1)^2}{4\cdot 4\cdot 3!}B_2B_2x^3-\frac{(2^6-1)^2}{6\cdot 6\cdot 5!}B_3B_3x^5-\cdots \quad (a=0)\\
&=\frac{(2^4-1)(2^2-1)}{2\cdot 4\cdot 1!}B_1B_2x+\frac{(2^6-1)(2^4-1)}{4\cdot 6\cdot 3!}B_2B_3x^3+\frac{(2^8-1)(2^6-1)}{6\cdot 8\cdot 5!}B_3B_4x^5+\cdots \quad (a=2)\\
&=-\frac{(2^6-1)(2^2-1)}{2\cdot 6\cdot 1!}B_1B_3x-\frac{(2^8-1)(2^4-1)}{4\cdot 8\cdot 3!}B_2B_4x^3-\frac{(2^{10}-1)(2^6-1)}{6\cdot 10\cdot 5!}B_3B_5x^5-\cdots \quad (a=4)\\
&\vdots\\
&=\frac{\ln 2}{2}-\frac{2^2-1}{2\cdot 2\cdot 1!}B_1x \quad (a=-1)\\
&=\frac{2^2-1}{2^3}\zeta(3)-\frac{(2^2-1)\pi^2}{2\cdot 1!\cdot 2!}B_1B_1x+\frac{(2^4-1)B_2}{2\cdot 4\cdot 3!}x^3 \quad (a=-3)\\
&=\frac{2^4-1}{2^5}\zeta(5)-\frac{(2^3-1)(2^2-1)\pi^4}{2\cdot 1!\cdot 4!}B_2B_1x+\frac{(2-1)(2^4-1)\pi^2}{4\cdot 3!\cdot 2!}B_1B_2x^3-\frac{(2^6-1)B_3}{2\cdot 6\cdot 5!}x^5 \quad (a=-5)\\
&=\frac{2^6-1}{2^7}\zeta(7)-\frac{(2^5-1)(2^2-1)\pi^6}{2\cdot 1!\cdot 6!}B_1B_3x+\frac{(2^3-1)(2^4-1)\pi^4}{4\cdot 3!\cdot 4!}B_2B_2x^3-\frac{(2-1)(2^6-1)\pi^2}{6\cdot 5!\cdot 2!}B_3B_1x^5\\
&\qquad +\frac{(2^8-1)B_4}{2\cdot 8\cdot 7!}x^7 \quad (a=-7)\\
&\vdots\\
&=\frac{\pi^2}{2\cdot 2!}B_1-\frac{(2^2-1)\ln 2}{2\cdot 1!}B_1x+\frac{(2^2-1)(2^4-1)}{2\cdot 4\cdot 3!}B_2B_1x^3+\frac{(2^4-1)(2^6-1)}{4\cdot 6\cdot 5!}B_3B_2x^5+\cdots \quad (a=-2)\\
&=\frac{(2^3-1)\pi^4}{2\cdot 4!}B_2-\frac{(2^2-1)(2^2-1)B_1}{2\cdot 2^2\cdot 1!}\zeta(3)x+\frac{(2^4-1)\ln 2}{4\cdot 3!}B_2x^3-\frac{(2^2-1)(2^6-1)}{2\cdot 6\cdot 5!}B_1B_3x^5\\
&\qquad -\frac{(2^4-1)(2^8-1)}{4\cdot 8\cdot 7!}B_2B_4x^7-\cdots \quad (a=-4)\\
&=\frac{(2^5-1)\pi^6}{2\cdot 6!}B_3-\frac{(2^4-1)(2^2-1)B_1}{2\cdot 2^4\cdot 1!}\zeta(5)x+\frac{(2^2-1)(2^4-1)B_2}{4\cdot 2^2\cdot 3!}\zeta(3)x^3-\frac{(2^6-1)\ln 2}{6\cdot 5!}B_3x^5+\frac{(2^2-1)(2^8-1)}{2\cdot 8\cdot 7!}B_1B_4x^7\\
&\qquad +\frac{(2^4-1)(2^{10}-1)}{4\cdot 10\cdot 9!}B_2B_5x^9+\cdots \quad (a=-6)\\
&\vdots
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
&\frac{1}{e^x+1}+\frac{3^a}{e^{3x}+1}+\frac{5^a}{e^{5x}+1}+\frac{7^a}{e^{7x}+1}+\cdots \\
&=\frac{\pi^2}{2\cdot 2}B_1\frac{1}{x^2}+\frac{B_1}{2\cdot 2}\quad (a=1) \\
&=\frac{(2^3-1)\pi^4}{2\cdot 4}B_2\frac{1}{x^4}-\frac{2^3-1}{2\cdot 4}B_2\quad (a=3) \\
&=\frac{(2^5-1)\pi^6}{2\cdot 6}B_3\frac{1}{x^6}+\frac{2^5-1}{2\cdot 6}B_3\quad (a=5) \\
&\vdots \\
&=\frac{\ln 2}{2}\frac{1}{x}-\frac{(2-1)(2^2-1)}{2\cdot 2\cdot 1!}B_1B_1x-\frac{(2^3-1)(2^4-1)}{4\cdot 4\cdot 3!}B_2B_2x^3-\cdots\quad (a=0) \\
&=\frac{(2^2-1)2!}{2^3}\zeta(3)\frac{1}{x^3}+\frac{(2^3-1)(2^2-1)}{2\cdot 4\cdot 1!}B_2B_1x+\frac{(2^5-1)(2^4-1)}{4\cdot 6\cdot 3!}B_3B_2x^3+\cdots\quad (a=2) \\
&=\frac{(2^4-1)4!}{2^5}\zeta(5)\frac{1}{x^5}-\frac{(2^5-1)(2^2-1)}{2\cdot 6\cdot 1!}B_3B_1x-\frac{(2^7-1)(2^4-1)}{4\cdot 8\cdot 3!}B_4B_2x^3-\cdots\quad (a=4) \\
&=\frac{(2^6-1)6!}{2^7}\zeta(7)\frac{1}{x^7}+\frac{(2^7-1)(2^2-1)}{2\cdot 8\cdot 1!}B_4B_1x+\frac{(2^9-1)(2^4-1)}{4\cdot 10\cdot 3!}B_5B_2x^3+\cdots\quad (a=6) \\
&\vdots \\
&=\frac{2^3-1}{2^4}\zeta(3)-\frac{(2^2-1)^2\pi^2}{2\cdot 2\cdot 1!\cdot 2!}B_1B_1x+\frac{2^3-1}{2^4\pi^2}\zeta(3)x^2\quad (a=-3) \\
&=\frac{2^5-1}{2^6}\zeta(5)-\frac{(2^4-1)(2^2-1)\pi^4}{2\cdot 2\cdot 1!\cdot 4!}B_2B_1x+\frac{(2^2-1)(2^4-1)\pi^2}{2\cdot 4\cdot 3!\cdot 2!}B_1B_2x^3-\frac{2^5-1}{2^6\pi^4}\zeta(5)x^4\quad (a=-5) \\
&=\frac{2^7-1}{2^8}\zeta(7)-\frac{(2^6-1)(2^2-1)\pi^6}{2\cdot 2\cdot 1!\cdot 6!}B_3B_1x+\frac{(2^4-1)(2^4-1)\pi^4}{2\cdot 4\cdot 3!\cdot 4!}B_2B_2x^3-\frac{(2^2-1)(2^6-1)\pi^2}{2\cdot 6\cdot 5!\cdot 2!}B_1B_3x^5 \\
&\qquad +\frac{2^7-1}{2^8\pi^6}\zeta(7)x^6\quad (a=-7) \\
&\vdots
\end{aligned} \tag{81}
$$

これらは $a=-1$ と $a=-2,-4,-6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^x+1}-\frac{3^a}{e^{3x}+1}+\frac{5^a}{e^{5x}+1}-\frac{7^a}{e^{7x}+1}+\cdots\\
&=\frac{2^2-1}{2\cdot 2\cdot 1!}E_1B_1x+\frac{2^4-1}{2\cdot 4\cdot 3!}E_2B_2x^3+\frac{2^6-1}{2\cdot 6\cdot 5!}E_3B_3x^5+\cdots \quad (a=1)\\
&=-\frac{2^2-1}{2\cdot 2\cdot 1!}B_1E_2x-\frac{2^4-1}{2\cdot 4\cdot 3!}B_2E_3x^3-\frac{2^6-1}{2\cdot 6\cdot 5!}B_3E_4x^5-\cdots \quad (a=3)\\
&=\frac{2^2-1}{2\cdot 2\cdot 1!}E_3B_1x+\frac{2^4-1}{2\cdot 4\cdot 3!}E_4B_2x^3+\frac{2^6-1}{2\cdot 6\cdot 5!}E_5B_3x^5+\cdots \quad (a=5)\\
&\vdots\\
&=\frac{E_0}{2\cdot 2} \quad (a=0)\\
&=-\frac{E_1}{2\cdot 2} \quad (a=2)\\
&=\frac{E_2}{2\cdot 2} \quad (a=4)\\
&=-\frac{E_3}{2\cdot 2} \quad (a=6)\\
&\vdots\\
&=\frac{\pi}{2^3\cdot 0!}E_0-\frac{2^2-1}{2\cdot 2\cdot 1!}B_1E_0x-\frac{2^4-1}{2\cdot 4\cdot 3!}B_2E_1x^3-\frac{2^6-1}{2\cdot 6\cdot 5!}B_3E_2x^5-\cdots \quad (a=-1)\\
&=\frac{\pi^3}{2^5\cdot 2!}E_1-\frac{2^2-1}{2\cdot 1!}B_1\mu(2)x+\frac{2^4-1}{2\cdot 4\cdot 3!}B_2E_0x^3+\frac{2^6-1}{2\cdot 6\cdot 5!}B_3E_1x^5+\cdots \quad (a=-3)\\
&=\frac{\pi^5}{2^7\cdot 4!}E_2-\frac{2^2-1}{2\cdot 1!}B_1\mu(4)x+\frac{2^4-1}{4\cdot 3!}B_2\mu(2)x^3-\frac{2^6-1}{2\cdot 6\cdot 5!}B_3E_0x^5-\frac{2^8-1}{2\cdot 8\cdot 7!}B_4E_1x^7-\cdots \quad (a=-5)\\
&\vdots\\
&=\frac{1}{2}\mu(2)-\frac{(2^2-1)\left(\frac{\pi}{2}\right)}{2\cdot 2\cdot 1!\cdot 0!}B_1E_0x \quad (a=-2)\\
&=\frac{1}{2}\mu(4)-\frac{(2^2-1)\left(\frac{\pi}{2}\right)^3}{2\cdot 2\cdot 1!\cdot 2!}B_1E_1x+\frac{(2^4-1)\left(\frac{\pi}{2}\right)}{2\cdot 4\cdot 3!\cdot 0!}B_2E_0x^3 \quad (a=-4)\\
&=\frac{1}{2}\mu(6)-\frac{(2^2-1)\left(\frac{\pi}{2}\right)^5}{2\cdot 2\cdot 1!\cdot 4!}B_1E_2x+\frac{(2^4-1)\left(\frac{\pi}{2}\right)^3}{2\cdot 4\cdot 3!\cdot 2!}B_2E_1x^3-\frac{(2^6-1)\left(\frac{\pi}{2}\right)}{2\cdot 6\cdot 5!\cdot 0!}B_3E_0x^5 \quad (a=-6)\\
&\vdots
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
&\frac{1}{e^x-e^{-x}}+\frac{2^a}{e^{2x}-e^{-2x}}+\frac{3^a}{e^{3x}-e^{-3x}}+\frac{4^a}{e^{4x}-e^{-4x}}+\frac{5^a}{e^{5x}-e^{-5x}}+\cdots\\
&=\frac{(2^2-1)\pi^2}{2\cdot 2}B_1\frac{1}{x^2}-\frac{1}{4x}\quad(a=1)\\
&=\frac{(2^4-1)\pi^4}{2\cdot 4}B_2\frac{1}{x^4}\quad(a=3)\\
&=\frac{(2^6-1)\pi^6}{2\cdot 6}B_3\frac{1}{x^6}\quad(a=5)\\
&=\frac{(2^8-1)\pi^8}{2\cdot 8}B_4\frac{1}{x^8}\quad(a=7)\\
&\vdots\\
&=\frac{(2^3-1)2!}{2^3}\zeta(3)\frac{1}{x^3}-\frac{B_1}{2\cdot 2}\frac{1}{x}-\frac{B_1B_2}{2\cdot 4\cdot 1!}x-\frac{(2^3-1)B_2B_3}{4\cdot 6\cdot 3!}x^3-\frac{(2^5-1)B_3B_4}{6\cdot 8\cdot 5!}x^5-\cdots\quad(a=2)\\
&=\frac{(2^5-1)4!}{2^5}\zeta(5)\frac{1}{x^5}+\frac{B_2}{2\cdot 4}\frac{1}{x}+\frac{B_1B_3}{2\cdot 6\cdot 1!}x+\frac{(2^3-1)B_2B_4}{4\cdot 8\cdot 3!}x^3+\frac{(2^5-1)B_3B_5}{6\cdot 10\cdot 5!}x^5+\cdots\quad(a=4)\\
&=\frac{(2^7-1)6!}{2^7}\zeta(7)\frac{1}{x^7}-\frac{B_3}{2\cdot 6}\frac{1}{x}-\frac{B_1B_4}{2\cdot 8\cdot 1!}x-\frac{(2^3-1)B_2B_5}{4\cdot 10\cdot 3!}x^3-\frac{(2^5-1)B_3B_6}{6\cdot 12\cdot 5!}x^5-\cdots\quad(a=6)\\
&\vdots\\
&=\frac{\pi^2}{2!}B_1\frac{1}{x}-\frac{\ln 2}{2}+\frac{B_1}{2\cdot 2\cdot 1!}x\quad(a=-1)\\
&=\frac{2^2\pi^4}{4!}B_2\frac{1}{x}-\frac{(2-1)(2\pi)^2}{2\cdot 2\cdot 1!\cdot 2!}B_1B_1x+\frac{2^2-1}{2(2\pi)^2}\zeta(3)x^2-\frac{2^3-1}{2\cdot 4\cdot 3!}B_2x^3\quad(a=-3)\\
&=\frac{2^4\pi^6}{6!}B_3\frac{1}{x}-\frac{(2-1)(2\pi)^4}{2\cdot 2\cdot 1!\cdot 4!}B_1B_2x+\frac{(2^3-1)(2\pi)^2}{2\cdot 4\cdot 3!\cdot 2!}B_2B_1x^3-\frac{2^4-1}{2(2\pi)^4}\zeta(5)x^4+\frac{2^5-1}{2\cdot 6\cdot 5!}B_3x^5\quad(a=-5)\\
&=\frac{2^6\pi^8}{8!}B_4\frac{1}{x}-\frac{(2-1)(2\pi)^6}{2\cdot 2\cdot 1!\cdot 6!}B_1B_3x+\frac{(2^3-1)(2\pi)^4}{2\cdot 4\cdot 3!\cdot 4!}B_2B_2x^3-\frac{(2^5-1)(2\pi)^2}{2\cdot 6\cdot 5\cdot 2!}B_3B_1x^5+\frac{2^6-1}{2(2\pi)^6}\zeta(7)x^6\\
&\qquad-\frac{2^7-1}{2\cdot 8\cdot 7!}B_4x^7\quad(a=-7)\\
&\vdots
\end{aligned}
\tag{83}
$$

これらは $a=0,-2,-4,-6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^x-e^{-x}}-\frac{2^a}{e^{2x}-e^{-2x}}+\frac{3^a}{e^{3x}-e^{-3x}}-\frac{4^a}{e^{4x}-e^{-4x}}+\cdots\\
&=\frac{\ln 2}{2}\frac{1}{x}-\frac{(2-1)(2^2-1)}{2\cdot 2\cdot 1!}B_1B_1x-\frac{(2^3-1)(2^4-1)}{4\cdot 4\cdot 3!}B_2B_2x^3-\frac{(2^5-1)(2^6-1)}{6\cdot 6\cdot 5!}B_3B_3x^5-\cdots\quad (a=0)\\
&=\frac{2^2-1}{2\cdot 2}B_1\frac{1}{x}+\frac{(2-1)(2^4-1)}{2\cdot 4\cdot 1!}B_1B_2x+\frac{(2^3-1)(2^6-1)}{4\cdot 6\cdot 3!}B_2B_3x^3+\frac{(2^5-1)(2^8-1)}{6\cdot 8\cdot 5!}B_3B_4x^5+\cdots\quad (a=2)\\
&=-\frac{2^4-1}{2\cdot 4}B_2\frac{1}{x}-\frac{(2-1)(2^6-1)}{2\cdot 6\cdot 1!}B_1B_3x-\frac{(2^3-1)(2^8-1)}{4\cdot 8\cdot 3!}B_2B_4x^3-\frac{(2^5-1)(2^{10}-1)}{6\cdot 10\cdot 5!}B_3B_5x^5-\cdots\quad (a=4)\\
&\vdots\\
&=\frac{1}{4x}\quad (a=1)\\
&=0\quad (a=3,5,7,9,\cdots)\\
&=\frac{\pi^2}{2\cdot 2!}B_1\frac{1}{x}-\frac{B_1}{2\cdot 2\cdot 1!}x\quad (a=-1)\\
&=\frac{(2^3-1)\pi^4}{2\cdot 4!}B_2\frac{1}{x}-\frac{(2-1)(2-1)\pi^2}{2\cdot 1!\cdot 2!}B_1B_1x+\frac{2^3-1}{2\cdot 4\cdot 3!}B_2x^3\quad (a=-3)\\
&=\frac{(2^5-1)\pi^6}{2\cdot 6!}B_3\frac{1}{x}-\frac{(2-1)(2^3-1)\pi^4}{2\cdot 1!\cdot 4!}B_1B_2x+\frac{(2^3-1)(2-1)\pi^2}{4\cdot 3!\cdot 2!}B_2B_1x^3-\frac{2^5-1}{2\cdot 6\cdot 5!}B_3x^5\quad (a=-5)\\
&=\frac{(2^7-1)\pi^8}{2\cdot 8!}B_4\frac{1}{x}-\frac{(2-1)(2^5-1)\pi^6}{2\cdot 1!\cdot 6!}B_1B_3x+\frac{(2^3-1)(2^3-1)\pi^4}{4\cdot 3!\cdot 4!}B_2B_2x^3-\frac{(2^5-1)(2-1)\pi^2}{6\cdot 5!\cdot 2!}B_3B_1x^5\\
&\qquad+\frac{2^7-1}{2\cdot 8\cdot 7!}B_4x^7\quad (a=-7)\\
&\vdots\\
&=\frac{2^2-1}{2^3}\zeta(3)\frac{1}{x}-\frac{(2-1)\ln 2}{2\cdot 1!}B_1x+\frac{(2^3-1)(2^2-1)}{2\cdot 4\cdot 3!}B_2B_1x^3+\frac{(2^5-1)(2^4-1)}{4\cdot 6\cdot 5!}B_3B_2x^5\\
&\qquad+\frac{(2^7-1)(2^6-1)}{6\cdot 8\cdot 7!}B_4B_3x^7+\cdots\quad (a=-2)\\
&=\frac{2^4-1}{2^5}\zeta(5)\frac{1}{x}-\frac{(2-1)(2^2-1)}{2^3\cdot 1!}B_1\zeta(3)x+\frac{(2^3-1)\ln 2}{4\cdot 3!}B_2x^3-\frac{(2^5-1)(2^2-1)}{2\cdot 6\cdot 5!}B_3B_1x^5\\
&\qquad-\frac{(2^7-1)(2^4-1)}{4\cdot 8\cdot 7!}B_4B_2x^7-\cdots\quad (a=-4)\\
&=\frac{2^6-1}{2^7}\zeta(7)\frac{1}{x}-\frac{(2-1)(2^4-1)}{2\cdot 2^4\cdot 1!}B_1\zeta(5)x+\frac{(2^3-1)(2^2-1)}{4\cdot 2^2\cdot 3!}B_2\zeta(3)x^3-\frac{(2^5-1)\ln 2}{6\cdot 5!}B_3x^5\\
&\qquad+\frac{(2^7-1)(2^2-1)}{2\cdot 8\cdot 7!}B_4B_1x^7+\frac{(2^9-1)(2^4-1)}{4\cdot 10\cdot 9!}B_5B_2x^9+\cdots\quad (a=-6)\\
&=\frac{2^8-1}{2^9}\zeta(9)\frac{1}{x}-\frac{(2-1)(2^6-1)}{2\cdot 2^6\cdot 1!}B_1\zeta(7)x+\frac{(2^3-1)(2^4-1)}{4\cdot 2^4\cdot 3!}B_2\zeta(5)x^3-\frac{(2^5-1)(2^2-1)}{6\cdot 2^2\cdot 5!}B_3\zeta(3)x^5\\
&\qquad+\frac{(2^7-1)\ln 2}{8\cdot 7!}B_4x^7-\frac{(2^9-1)(2^2-1)}{2\cdot 10\cdot 9!}B_5B_1x^9-\cdots\quad (a=-8)\\
&\vdots
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&\frac{1}{e^{x}-e^{-x}}+\frac{3^{a}}{e^{3x}-e^{-3x}}+\frac{5^{a}}{e^{5x}-e^{-5x}}+\frac{7^{a}}{e^{7x}-e^{-7x}}+\cdots\\
&=\frac{(2^{2}-1)\pi^{2}}{4\cdot 2}B_{1}\frac{1}{x^{2}}\quad (a=1)\\
&=\frac{(2^{4}-1)\pi^{4}}{4\cdot 4}B_{2}\frac{1}{x^{4}}\quad (a=3)\\
&=\frac{(2^{6}-1)\pi^{6}}{4\cdot 6}B_{3}\frac{1}{x^{6}}\quad (a=5)\\
&\vdots\\
&=\frac{(2^{3}-1)2!}{2^{4}}\zeta(3)\frac{1}{x^{3}}+\frac{B_{1}}{2\cdot 2}\frac{1}{x}+\frac{(2-1)(2^{3}-1)}{2\cdot 4\cdot 1!}B_{1}B_{2}x+\frac{(2^{3}-1)(2^{5}-1)}{4\cdot 6\cdot 3!}B_{2}B_{3}x^{3}\\
&\qquad\qquad+\frac{(2^{5}-1)(2^{7}-1)}{6\cdot 8\cdot 5!}B_{3}B_{4}x^{5}+\cdots\quad (a=2)\\
&=\frac{(2^{5}-1)4!}{2^{6}}\zeta(5)\frac{1}{x^{5}}-\frac{2^{3}-1}{2\cdot 4}B_{2}\frac{1}{x}-\frac{(2-1)(2^{5}-1)}{2\cdot 6\cdot 1!}B_{1}B_{3}x-\frac{(2^{3}-1)(2^{7}-1)}{4\cdot 8\cdot 3!}B_{2}B_{4}x^{3}\\
&\qquad\qquad-\frac{(2^{5}-1)(2^{9}-1)}{6\cdot 10\cdot 5!}B_{3}B_{5}x^{5}-\cdots\quad (a=4)\\
&=\frac{(2^{7}-1)6!}{2^{8}}\zeta(7)\frac{1}{x^{7}}+\frac{2^{5}-1}{2\cdot 6}B_{3}\frac{1}{x}+\frac{(2-1)(2^{7}-1)}{2\cdot 8\cdot 1!}B_{1}B_{4}x+\frac{(2^{3}-1)(2^{9}-1)}{4\cdot 10\cdot 3!}B_{2}B_{5}x^{3}\\
&\qquad\qquad+\frac{(2^{5}-1)(2^{11}-1)}{6\cdot 12\cdot 5!}B_{3}B_{6}x^{5}+\cdots\quad (a=6)\\
&\vdots\\
&=\frac{(2^{2}-1)\pi^{2}}{4\cdot 2!}B_{1}\frac{1}{x}-\frac{\ln 2}{4}\quad (a=-1)\\
&=\frac{(2^{4}-1)\pi^{4}}{4\cdot 4!}B_{2}\frac{1}{x}-\frac{(2-1)(2^{2}-1)\pi^{2}}{2\cdot 2\cdot 1!\cdot 2!}B_{1}B_{1}x+\frac{2^{2}-1}{2^{4}\pi^{2}}\zeta(3)x^{2}\quad (a=-3)\\
&=\frac{(2^{6}-1)\pi^{6}}{4\cdot 6!}B_{3}\frac{1}{x}-\frac{(2-1)(2^{4}-1)\pi^{4}}{2\cdot 2\cdot 1!\cdot 4!}B_{1}B_{2}x+\frac{(2^{3}-1)(2^{2}-1)\pi^{2}}{2\cdot 4\cdot 3!\cdot 2!}B_{2}B_{1}x^{3}-\frac{2^{4}-1}{2^{6}\pi^{4}}\zeta(5)x^{4}\quad (a=-5)\\
&=\frac{(2^{8}-1)\pi^{8}}{4\cdot 8!}B_{4}\frac{1}{x}-\frac{(2-1)(2^{6}-1)\pi^{6}}{2\cdot 2\cdot 1!\cdot 6!}B_{1}B_{3}x+\frac{(2^{3}-1)(2^{4}-1)\pi^{4}}{2\cdot 4\cdot 3!\cdot 4!}B_{2}B_{2}x^{3}-\frac{(2^{5}-1)(2^{2}-1)\pi^{2}}{2\cdot 6\cdot 5!\cdot 2!}B_{3}B_{1}x^{5}\\
&\qquad\qquad+\frac{2^{6}-1}{2^{8}\pi^{6}}\zeta(7)x^{6}\quad (a=-7)\\
&\vdots
\end{aligned}
\tag{85}
$$

これらは $a=0,-2,-4,-6,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^{x}-e^{-x}}-\frac{3^{a}}{e^{3x}-e^{-3x}}+\frac{5^{a}}{e^{5x}-e^{-5x}}-\frac{7^{a}}{e^{7x}-e^{-7x}}+\cdots\\
&=\frac{E_0}{4}\frac{1}{x}+\frac{2-1}{2\cdot 2\cdot 1!}B_1E_1x+\frac{2^3-1}{2\cdot 4\cdot 3!}B_2E_2x^3+\frac{2^5-1}{2\cdot 6\cdot 5!}B_3E_3x^5+\cdots \quad (a=1)\\
&=-\frac{E_1}{4}\frac{1}{x}-\frac{2-1}{2\cdot 2\cdot 1!}B_1E_2x-\frac{2^3-1}{2\cdot 4\cdot 3!}B_2E_3x^3-\frac{2^5-1}{2\cdot 6\cdot 5!}B_3E_4x^5-\cdots \quad (a=3)\\
&=\frac{E_2}{4}\frac{1}{x}+\frac{2-1}{2\cdot 2\cdot 1!}B_1E_3x+\frac{2^3-1}{2\cdot 4\cdot 3!}B_2E_4x^3+\frac{2^5-1}{2\cdot 6\cdot 5!}B_3E_5x^5+\cdots \quad (a=5)\\
&\vdots\\
&=\frac{\pi}{8}\frac{1}{x} \quad (a=0)\\
&=0 \quad (a=2,4,6,8,\cdots)\\
&=\frac{1}{2}\mu(2)\frac{1}{x}-\frac{2-1}{2\cdot 2\cdot 1!}B_1E_0x-\frac{2^3-1}{2\cdot 4\cdot 3!}B_2E_1x^3-\frac{2^5-1}{2\cdot 6\cdot 5!}B_3E_2x^5-\cdots \quad (a=-1)\\
&=\frac{1}{2}\mu(4)\frac{1}{x}-\frac{2-1}{2\cdot 1!}B_1\mu(2)x+\frac{2^3-1}{2\cdot 4\cdot 3!}B_2E_0x^3+\frac{2^5-1}{2\cdot 6\cdot 5!}B_3E_1x^5+\cdots \quad (a=-3)\\
&=\frac{1}{2}\mu(6)\frac{1}{x}-\frac{2-1}{2\cdot 1!}B_1\mu(4)x+\frac{2^3-1}{4\cdot 3!}B_2\mu(2)x^3-\frac{2^5-1}{2\cdot 6\cdot 5!}B_3E_0x^5-\cdots \quad (a=-5)\\
&\vdots\\
&=\frac{\left(\frac{\pi}{2}\right)^3}{4\cdot 2!}E_1\frac{1}{x}-\frac{(2-1)\left(\frac{\pi}{2}\right)}{2\cdot 2\cdot 0!\cdot 1!}B_1E_0x \quad (a=-2)\\
&=\frac{\left(\frac{\pi}{2}\right)^5}{4\cdot 4!}E_2\frac{1}{x}-\frac{(2-1)\left(\frac{\pi}{2}\right)^3}{2\cdot 2\cdot 1!\cdot 2!}B_1E_1x+\frac{(2^3-1)\left(\frac{\pi}{2}\right)}{2\cdot 4\cdot 3!\cdot 0!}B_2E_0x^3 \quad (a=-4)\\
&=\frac{\left(\frac{\pi}{2}\right)^7}{4\cdot 6!}E_3\frac{1}{x}-\frac{(2-1)\left(\frac{\pi}{2}\right)^5}{2\cdot 2\cdot 1!\cdot 4!}B_1E_2x+\frac{(2^3-1)\left(\frac{\pi}{2}\right)^3}{2\cdot 4\cdot 3!\cdot 2!}B_2E_1x^3-\frac{(2^5-1)\left(\frac{\pi}{2}\right)}{2\cdot 6\cdot 5!\cdot 0!}B_3E_0x^5 \quad (a=-6)\\
&=\frac{\left(\frac{\pi}{2}\right)^9}{4\cdot 8!}E_4\frac{1}{x}-\frac{(2-1)\left(\frac{\pi}{2}\right)^7}{2\cdot 2\cdot 1!\cdot 6!}B_1E_3x+\frac{(2^3-1)\left(\frac{\pi}{2}\right)^5}{2\cdot 4\cdot 3!\cdot 4!}B_2E_2x^3-\frac{(2^5-1)\left(\frac{\pi}{2}\right)^3}{2\cdot 6\cdot 5!\cdot 2!}B_3E_1x^5+\frac{(2^7-1)\left(\frac{\pi}{2}\right)}{2\cdot 8\cdot 7!\cdot 0!}B_4E_0x^7 \quad (a=-8)\\
&\vdots
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
&\frac{1}{e^{x}+e^{-x}}+\frac{2^{a}}{e^{2x}+e^{-2x}}+\frac{3^{a}}{e^{3x}+e^{-3x}}+\frac{4^{a}}{e^{4x}+e^{-4x}}+\frac{5^{a}}{e^{5x}+e^{-5x}}+\cdots\\
&=1!\cdot\mu(2)\frac{1}{x^{2}}-\frac{E_0B_1}{2\cdot 2\cdot 0!}-\frac{E_1B_2}{2\cdot 4\cdot 2!}x^{2}-\frac{E_2B_3}{2\cdot 6\cdot 4!}x^{4}-\cdots\quad(a=1)\\
&=3!\cdot\mu(4)\frac{1}{x^{4}}+\frac{E_0B_2}{2\cdot 4\cdot 0!}+\frac{E_1B_3}{2\cdot 6\cdot 2!}x^{2}+\frac{E_2B_4}{2\cdot 8\cdot 4!}x^{4}+\cdots\quad(a=3)\\
&=5!\cdot\mu(6)\frac{1}{x^{6}}-\frac{E_0B_3}{2\cdot 6\cdot 0!}-\frac{E_1B_4}{2\cdot 8\cdot 2!}x^{2}-\frac{E_2B_5}{2\cdot 10\cdot 4!}x^{4}-\cdots\quad(a=5)\\
&\vdots\\
&=\frac{1}{2}\left(\frac{\pi}{2}\right)E_0\frac{1}{x}-\frac{1}{4}\quad(a=0)\\
&=\frac{1}{2}\left(\frac{\pi}{2}\right)^{3}E_1\frac{1}{x^{3}}\quad(a=2)\\
&=\frac{1}{2}\left(\frac{\pi}{2}\right)^{5}E_2\frac{1}{x^{5}}\quad(a=4)\\
&=\frac{1}{2}\left(\frac{\pi}{2}\right)^{7}E_3\frac{1}{x^{7}}\quad(a=6)\\
&\vdots\\
&=\frac{(2\pi)^{2}}{2\cdot 2\cdot 0!\cdot 2!}E_0B_1-\frac{2}{\pi}\mu(2)x+\frac{E_1}{2\cdot 2\cdot 2!}x^{2}\quad(a=-2)\\
&=\frac{(2\pi)^{4}}{2\cdot 2\cdot 0!\cdot 4!}E_0B_2-\frac{(2\pi)^{2}}{2\cdot 2\cdot 2!\cdot 2!}E_1B_1x^{2}+\left(\frac{2}{\pi}\right)^{3}\mu(4)x^{3}-\frac{E_2}{2\cdot 2\cdot 4!}x^{4}\quad(a=-2)\\
&=\frac{(2\pi)^{6}}{2\cdot 2\cdot 0!\cdot 6!}E_0B_3-\frac{(2\pi)^{4}}{2\cdot 2\cdot 2!\cdot 4!}E_1B_2x^{2}+\frac{(2\pi)^{2}}{2\cdot 2\cdot 4!\cdot 2!}E_2B_1x^{4}-\left(\frac{2}{\pi}\right)^{5}\mu(6)x^{5}+\frac{E_3}{2\cdot 2\cdot 6!}x^{6}\quad(a=-6)\\
&\vdots
\end{aligned}
\tag{87}
$$

これらは $a=-1,-3,-5,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^{x}+e^{-x}}-\frac{2^{a}}{e^{2x}+e^{-2x}}+\frac{3^{a}}{e^{3x}+e^{-3x}}-\frac{4^{a}}{e^{4x}+e^{-4x}}+\frac{5^{a}}{e^{5x}+e^{-5x}}-\cdots\\
&=\frac{2^{2}-1}{2\cdot 2\cdot 0!}E_0B_1+\frac{2^{4}-1}{2\cdot 4\cdot 2!}E_1B_2x^2+\frac{2^{6}-1}{2\cdot 6\cdot 4!}E_2B_3x^4+\cdots \quad (a=1)\\
&=-\frac{2^{4}-1}{2\cdot 4\cdot 0!}E_0B_2-\frac{2^{6}-1}{2\cdot 6\cdot 2!}E_1B_3x^2-\frac{2^{8}-1}{2\cdot 8\cdot 4!}E_2B_4x^4-\cdots \quad (a=3)\\
&=\frac{2^{6}-1}{2\cdot 6\cdot 0!}E_0B_3+\frac{2^{8}-1}{2\cdot 8\cdot 2!}E_1B_4x^2+\frac{2^{10}-1}{2\cdot 10\cdot 4!}E_2B_5x^4+\cdots \quad (a=5)\\
&\vdots\\
&=\frac{1}{4} \quad (a=0)\\
&=0 \quad (a=2,4,6,8,10,\cdots)\\
&=\frac{\ln 2}{2}-\frac{2^{2}-1}{2\cdot 2\cdot 2!}E_1B_1x^2-\frac{2^{4}-1}{2\cdot 4\cdot 4!}E_2B_2x^4-\frac{2^{6}-1}{2\cdot 6\cdot 6!}E_3B_3x^6-\cdots \quad (a=-1)\\
&=\frac{2^{2}-1}{2^{3}}E_0\zeta(3)-\frac{\ln 2}{2\cdot 2!}E_1x^2+\frac{2^{2}-1}{2\cdot 2\cdot 4!}E_2B_1x^4+\frac{2^{4}-1}{2\cdot 4\cdot 6!}E_3B_2x^6+\cdots \quad (a=-3)\\
&=\frac{2^{4}-1}{2^{5}}E_0\zeta(5)-\frac{2^{2}-1}{2^{3}\cdot 2!}E_1\zeta(3)x^2+\frac{\ln 2}{2\cdot 4!}E_2x^4-\frac{2^{2}-1}{2\cdot 2\cdot 6!}E_3B_1x^6-\cdots \quad (a=-5)\\
&\vdots\\
&=\frac{\pi^2}{2\cdot 0!\cdot 2!}E_0B_1-\frac{E_1}{4\cdot 2!}x^2 \quad (a=-2)\\
&=\frac{(2^{3}-1)\pi^4}{2\cdot 0!\cdot 4!}E_0B_2-\frac{\pi^2}{2\cdot 2!\cdot 2!}E_1B_1x^2+\frac{E_2}{4\cdot 4!}x^4 \quad (a=-4)\\
&=\frac{(2^{5}-1)\pi^6}{2\cdot 0!\cdot 6!}E_0B_3-\frac{(2^{3}-1)\pi^4}{2\cdot 2!\cdot 4!}E_1B_2x^2+\frac{\pi^2}{2\cdot 4!\cdot 2!}E_2B_1x^4-\frac{E_3}{4\cdot 6!}x^6 \quad (a=-6)\\
&=\frac{(2^{7}-1)\pi^8}{2\cdot 0!\cdot 8!}E_0B_4-\frac{(2^{5}-1)\pi^6}{2\cdot 2!\cdot 6!}E_1B_3x^2+\frac{(2^{3}-1)\pi^4}{2\cdot 4!\cdot 4!}E_2B_2x^4-\frac{\pi^2}{2\cdot 6!\cdot 2!}E_3B_1x^6+\frac{E_4}{4\cdot 8!}x^8 \quad (a=-8)\\
&\vdots
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
&\frac{1}{e^{x}+e^{-x}}+\frac{3^{a}}{e^{3x}+e^{-3x}}+\frac{5^{a}}{e^{5x}+e^{-5x}}+\frac{7^{a}}{e^{7x}+e^{-7x}}+\frac{9^{a}}{e^{9x}+e^{-9x}}+\cdots \\
&=\frac{1}{2}\mu(2)\frac{1}{x^{2}}+\frac{2-1}{2\cdot 2\cdot 0!}E_0B_1+\frac{2^{3}-1}{2\cdot 4\cdot 2!}E_1B_2x^{2}+\frac{2^{5}-1}{2\cdot 6\cdot 4!}E_2B_3x^{4}+\cdots \quad (a=1) \\
&=\frac{3!}{2}\mu(4)\frac{1}{x^{4}}-\frac{2^{3}-1}{2\cdot 4\cdot 0!}E_0B_2-\frac{2^{5}-1}{2\cdot 6\cdot 2!}E_1B_3x^{2}-\frac{2^{7}-1}{2\cdot 8\cdot 4!}E_2B_4x^{4}-\cdots \quad (a=3) \\
&=\frac{5!}{2}\mu(6)\frac{1}{x^{6}}+\frac{2^{5}-1}{2\cdot 6\cdot 0!}E_0B_3+\frac{2^{7}-1}{2\cdot 8\cdot 2!}E_1B_4x^{2}+\frac{2^{9}-1}{2\cdot 10\cdot 4!}E_2B_5x^{4}+\cdots \quad (a=5) \\
&\vdots \\
&=\frac{1}{4}\left(\frac{\pi}{2}\right)E_0\frac{1}{x} \quad (a=0) \\
&=\frac{1}{4}\left(\frac{\pi}{2}\right)^{3}E_1\frac{1}{x^{3}} \quad (a=2) \\
&=\frac{1}{4}\left(\frac{\pi}{2}\right)^{5}E_2\frac{1}{x^{5}} \quad (a=4) \\
&\vdots \\
&=\frac{(2^{2}-1)\pi^{2}}{2\cdot 2\cdot 0!\cdot 2!}E_0B_1-\frac{1}{2}\left(\frac{2}{\pi}\right)\mu(2)x \quad (a=-2) \\
&=\frac{(2^{4}-1)\pi^{4}}{2\cdot 2\cdot 0!\cdot 4!}E_0B_2-\frac{(2^{2}-1)\pi^{2}}{2\cdot 2\cdot 2!\cdot 2!}E_1B_1x^{2}+\frac{1}{2}\left(\frac{2}{\pi}\right)^{3}\mu(4)x^{3} \quad (a=-4) \\
&=\frac{(2^{6}-1)\pi^{6}}{2\cdot 2\cdot 0!\cdot 6!}E_0B_3-\frac{(2^{4}-1)\pi^{4}}{2\cdot 2\cdot 2!\cdot 4!}E_1B_2x^{2}+\frac{(2^{2}-1)\pi^{2}}{2\cdot 2\cdot 4!\cdot 2!}E_2B_1x^{4}-\frac{1}{2}\left(\frac{2}{\pi}\right)^{5}\mu(6)x^{5} \quad (a=-6) \\
&\vdots
\end{aligned} \tag{89}
$$

これらは $a=-1,-3,-5,\cdots$ で係数が発散する。

$$
\begin{aligned}
&\frac{1}{e^x+e^{-x}}-\frac{3^a}{e^{3x}+e^{-3x}}+\frac{5^a}{e^{5x}+e^{-5x}}-\frac{7^a}{e^{7x}+e^{-7x}}+\frac{9^a}{e^{9x}+e^{-9x}}-\cdots\\
&=\frac{E_0E_0}{4\cdot 0!}+\frac{E_1E_1}{4\cdot 2!}x^2+\frac{E_2E_2}{4\cdot 4!}x^4+\frac{E_3E_3}{4\cdot 6!}x^6+\cdots\quad(a=0)\\
&=-\frac{E_0E_1}{4\cdot 0!}-\frac{E_1E_2}{4\cdot 2!}x^2-\frac{E_2E_3}{4\cdot 4!}x^4-\frac{E_3E_4}{4\cdot 6!}x^6-\cdots\quad(a=2)\\
&=\frac{E_0E_2}{4\cdot 0!}+\frac{E_1E_3}{4\cdot 2!}x^2+\frac{E_2E_4}{4\cdot 4!}x^4+\frac{E_3E_5}{4\cdot 6!}x^6+\cdots\quad(a=4)\\
&\vdots\\
&=\frac{\left(\frac{\pi}{2}\right)E_0E_0}{2\cdot 2\cdot 0!\cdot 0!}\quad(a=-1)\\
&=\frac{\left(\frac{\pi}{2}\right)^3E_0E_1}{2\cdot 2\cdot 0!\cdot 2!}-\frac{\left(\frac{\pi}{2}\right)E_1E_0}{2\cdot 2\cdot 2!\cdot 0!}x^2\quad(a=-1)\\
&=\frac{\left(\frac{\pi}{2}\right)^5E_0E_2}{2\cdot 2\cdot 0!\cdot 4!}-\frac{\left(\frac{\pi}{2}\right)^3E_1E_1}{2\cdot 2\cdot 2!\cdot 2!}x^2+\frac{\left(\frac{\pi}{2}\right)E_2E_0}{2\cdot 2\cdot 4!\cdot 0!}x^4\quad(a=-5)\\
&=\frac{\left(\frac{\pi}{2}\right)^7E_0E_3}{2\cdot 2\cdot 0!\cdot 6!}-\frac{\left(\frac{\pi}{2}\right)^5E_1E_2}{2\cdot 2\cdot 2!\cdot 4!}x^2+\frac{\left(\frac{\pi}{2}\right)^3E_2E_1}{2\cdot 2\cdot 4!\cdot 2!}x^4-\frac{\left(\frac{\pi}{2}\right)E_3E_0}{2\cdot 2\cdot 6!\cdot 0!}x^6\quad(a=-7)\\
&\vdots\\
&=\frac{E_0}{2\cdot 0!}\mu(2)-\frac{E_1E_0}{4\cdot 2!}x^2-\frac{E_2E_1}{4\cdot 4!}x^4-\frac{E_3E_2}{4\cdot 6!}x^6-\cdots\quad(a=-2)\\
&=\frac{E_0}{2\cdot 0!}\mu(4)-\frac{E_1}{2\cdot 2!}\mu(2)x^2+\frac{E_2E_0}{4\cdot 4!}x^4+\frac{E_3E_1}{4\cdot 6!}x^6+\cdots\quad(a=-4)\\
&=\frac{E_0}{2\cdot 0!}\mu(6)-\frac{E_1}{2\cdot 2!}\mu(4)x^2+\frac{E_2}{2\cdot 4!}\mu(2)x^4-\frac{E_3E_0}{4\cdot 6!}x^6-\cdots\quad(a=-6)\\
&\vdots\\
&=0\quad(a=1,3,5,7,\cdots)
\end{aligned}
\tag{90}
$$