

公式集

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2025 年 3 月 22 日

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1 INTRODUCTION

学生のときにこういう方面に凄くはまっていた時期がありました。その時期に導出した公式とかをノートにまとめたものです。あまり厳密なことは気にせずにやってたのであしからずです。

2 積分表示された関数の級数表示の 1 例

様々な積分表示された関数を級数で表すこと。

$$\begin{aligned}\int_0^1 \frac{(1-x)^{s-1}}{(1-ux)^{\alpha+1}} x^{\beta-1} dx &= \int_0^1 (1-x)^{s-1} x^{\beta-1} \sum_{n=0}^{\infty} \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} u^n x^n dx \\ &= \sum_{n=0}^{\infty} \frac{(\alpha+1)\cdots(\alpha+n)}{n!} u^n \int_0^1 (1-x)^{s-1} x^{\beta+n-1} dx \\ &= \sum_{n=0}^{\infty} \frac{(\alpha+1)\cdots(\alpha+n)}{n!} \frac{\beta(\beta+1)\cdots(\beta+n-1)}{(s+\beta)(s+\beta+1)\cdots(s+\beta+n-1)} u^n \frac{\Gamma(s)\Gamma(\beta)}{\Gamma(s+\beta)}\end{aligned}\quad (1)$$

これに $u \rightarrow 1$ の極限を取ればまとめると

$$\frac{\Gamma(s+\beta)\Gamma(s-\alpha)}{\Gamma(s)\Gamma(s-\alpha+\beta)} = 1 + \frac{\alpha\beta}{1!(s+\beta)} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!(s+\beta)(s+\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!(s+\beta)(s+\beta+1)(s+\beta+2)} + \cdots\quad (2)$$

を得る。同様な変形をして

$$\int_0^1 x^{s-1}(1-ux)^{\beta-1}dx = \frac{1}{s} - \frac{\beta-1}{1!(s+1)}u + \frac{(\beta-1)(\beta-2)}{2!(s+2)}u^2 - \frac{(\beta-1)(\beta-2)(\beta-3)}{3!(s+3)}u^3 + \dots \quad (3)$$

よって $u \rightarrow 1$ の極限を取れば

$$\frac{\Gamma(s)\Gamma(\beta)}{\Gamma(s+\beta)} = \frac{1}{s} - \frac{(\beta-1)}{1!(s+1)} + \frac{(\beta-1)(\beta-2)}{2!(s+2)} - \frac{(\beta-1)(\beta-2)(\beta-3)}{3!(s+3)} + \dots \quad (4)$$

などを得る。例えば $s = \beta = \frac{\pi}{2}$ を代入して

$$\pi = 2 + \frac{1}{1! \cdot 3} + \frac{1 \cdot 3}{2! \cdot 5 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{3! \cdot 7 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4! \cdot 9 \cdot 2^3} + \dots \quad (5)$$

などが得られる。また $\beta = 1 - s$ を代入すれば

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} + \frac{s}{1!(s+1)} + \frac{s(s+1)}{2!(s+2)} + \frac{s(s+1)(s+2)}{3!(s+3)} + \dots \quad (6)$$

を得る。etc...

3 オイラー・マクローリンの公式

関数 $F(x+h)$ のテイラー展開は形式的には

$$\begin{aligned} F(x+h) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d}{dx} \right)^n F(x) \\ &= \exp \left(h \frac{d}{dx} \right) F(x) \end{aligned} \quad (7)$$

と書ける。このことに留意して

$$\begin{aligned} F(x) + F(x+h) + F(x+2h) + \dots + F(x+Nh) &= \sum_{n=0}^N e^{nh \frac{d}{dx}} F(x) \\ &= \frac{1 - e^{(N+1)h \frac{d}{dx}}}{1 - e^{h \frac{d}{dx}}} F(x) \end{aligned} \quad (8)$$

と形式的に書ける。ここで”複素解析のお話”noteによれば

$$\frac{x}{1-e^x} = -1 + \frac{1}{2}x - \frac{B_1}{2!}x^2 + \frac{B_2}{4!}x^4 - \frac{B_3}{6!}x^6 + \dots \quad (9)$$

とかけた。従ってこの式に形式的に x に $\frac{d}{dx}$ を”代入”すると、 $\frac{dF}{dx}(x) = f(x)$ と置けば

$$\begin{aligned} \sum_{n=0}^N f(x+nh) &= \int_x^{x+(N+1)h} f(y)dy - \frac{1}{2} \left(f(x+(N+1)h) - f(x) \right) + \frac{B_1}{2!} \frac{d}{dx} \left(f(x+(N+1)h) - f(x) \right) \\ &\quad - \frac{B_2}{4!} \frac{d^3}{dx^3} \left(f(x+(N+1)h) - f(x) \right) + \frac{B_3}{6!} \frac{d^5}{dx^5} \left(f(x+(N+1)h) - f(x) \right) - \dots \end{aligned} \quad (10)$$

となる。これをオイラー・マクローリンの公式という。これを用いて簡単な例を計算してみる。

まずオイラー・マクローリンの公式を用いれば

$$S^k(N) := \sum_{n=1}^N n^k \quad (11)$$

が計算できる。 $f(x) = x^k$ と置き、最後に $x = 0$ 、 $h = 1$ と置けばよい。計算すると

$$S^k(N) = \frac{(N+1)^{k+1}}{k+1} - \frac{1}{2}(N+1)^k + \frac{B_1}{2!}k(N+1)^{k-1} - \frac{B_2}{4!}k(k-1)(k-2)(N+1)^{k-3} + \dots \quad (12)$$

となる。ここで和は $N+1$ の 1 次以上の項までとる。 $k = 1$ の時は

$$S^1(N) = \frac{N(N+1)}{2} \quad (13)$$

$k = 2$ の時は

$$S^2(N) = \frac{N(N+1)(2N+1)}{6} \quad (14)$$

などよく知られた公式が得られる。

他にも応用例はあるが、ここでは省略する。また工夫すれば $\sum_n (-1)^n f(n)$ のようなものも同様な計算をして公式を得ることが出来るが、それも省略する。

4 冪級数表示の 1 例

”解析の話発展編”note によれば

$$\int_0^\infty f(x)x^{s-1}dx = \Gamma(s)\phi(s) \quad (15)$$

の時

$$f(x) = \sum_{n=0}^\infty (-1)^n \frac{\phi(-n)}{\Gamma(1+n)} x^n \quad (16)$$

となる。この関係式を使って得られるいくつかの公式

$$\begin{aligned} \sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x \dots &= \frac{1}{2} \cot \frac{x}{2} \\ \sin x - \sin 2x + \sin 3x - \sin 4x + \dots &= \frac{B_1}{2!}(2^2-1)x + \frac{B_2}{4!}(2^4-1)x^3 + \frac{B_3}{6!}(2^6-1)x^5 + \dots \\ &= \frac{1}{2} \tan \frac{x}{2} \end{aligned} \quad (17)$$

他には

$$\frac{e^x}{(e^x-1)^2} = \frac{1}{x^2} - \frac{1}{2} \frac{B_1}{0!} + \frac{1}{4} \frac{B_2}{2!} x^2 - \frac{1}{6} \frac{B_3}{4!} x^4 + \dots \quad (18)$$

など得られる。これは実際に $\frac{d}{dx}(\frac{1}{e^x-1}) = \frac{-e^x}{(e^x-1)^2}$ を用いて計算した結果と一致する。この調子で計算した結果を列挙していく。

$$\begin{aligned} &\sin x + \frac{1}{2^a} \sin 2x + \frac{1}{3^a} \sin 3x + \frac{1}{4^a} \sin 4x + \dots \\ &= \frac{\pi}{2} - \frac{1}{2} x \quad (a=1) \\ &= \pi^2 B_1 x - \frac{\pi}{4} x^2 + \frac{1}{3! \cdot 2} x^3 \quad (a=3) \\ &= \frac{2^3 \pi^4}{4!} B_2 x - \frac{\pi^2}{3! \cdot 6} x^3 + \frac{\pi}{4! \cdot 2} x^4 - \frac{1}{2 \cdot 5!} x^5 \quad (a=5) \\ &= \frac{2^5 \pi^6}{6!} B_3 x - \frac{2^3 \pi^4}{3! \cdot 4!} B_2 x^3 + \frac{2\pi^2}{5! \cdot 2!} B_1 x^5 - \frac{\pi}{6! \cdot 2} x^6 + \frac{1}{2 \cdot 7!} x^7 \quad (a=7) \\ &= \frac{2^7 \pi^8}{8!} B_4 x - \frac{2^5 \pi^6}{3! \cdot 6!} B_3 x^3 + \frac{2^3 \pi^4}{5! \cdot 4!} B_2 x^5 - \frac{2\pi^2}{7! \cdot 2!} B_1 x^7 + \frac{\pi}{8! \cdot 2} x^8 - \frac{1}{2 \cdot 9!} x^9 \quad (a=9) \\ &\vdots \end{aligned} \quad (19)$$

これらは a が偶数の場合には係数が発散する。 $a = -1, -3, -5, \dots$ の場合には $= 0$ となる。

$$\begin{aligned}
& \sin x - \frac{1}{2^a} \sin 2x + \frac{1}{3^a} \sin 3x - \frac{1}{4^a} \sin 4x + \dots \\
&= \frac{1}{2} x \quad (a = 1) \\
&= \ln 2 \cdot x + (1 - 2^2) \frac{B_1}{2 \cdot 3!} x^3 + (1 - 2^4) \frac{B_2}{4 \cdot 5!} x^5 + (1 - 2^6) \frac{B_3}{6 \cdot 7!} x^7 + \dots \quad (a = 2) \\
&= \frac{3}{4} \zeta(3) x - \frac{\ln 2}{6} x^3 - (1 - 2^2) \frac{B_1}{5! \cdot 2} x^5 - (1 - 2^4) \frac{B_2}{7! \cdot 4} x^7 - (1 - 2^6) \frac{B_3}{9! \cdot 6} x^9 - \dots \quad (a = 4) \\
&= \frac{15}{16} \zeta(5) x - \frac{1}{8} \zeta(3) x^3 + \frac{\ln 2}{5!} x^5 + (1 - 2^2) \frac{B_1}{7! \cdot 2} x^7 + (1 - 2^4) \frac{B_2}{9! \cdot 4} x^9 + \dots \quad (a = 6) \\
&= \frac{2^6 - 1}{2^6} \zeta(7) x - \frac{2^4 - 1}{3! \cdot 2^4} \zeta(5) x^3 + \frac{2^2 - 1}{5! \cdot 2^2} \zeta(3) x^5 - \frac{\ln 2}{7!} x^7 - (1 - 2^2) \frac{B_1}{9! \cdot 2} x^9 - (1 - 2^4) \frac{B_2}{11! \cdot 4} x^{11} - \dots \quad (a = 8) \\
&= \frac{2^8 - 1}{2^8} \zeta(9) x - \frac{2^6 - 1}{3! \cdot 2^6} \zeta(7) x^3 + \frac{2^4 - 1}{5! \cdot 2^4} \zeta(5) x^5 - \frac{2^2 - 1}{7! \cdot 2^2} \zeta(3) x^7 + \frac{\ln 2}{9!} x^9 + (1 - 2^2) \frac{B_1}{11! \cdot 2} x^{11} \\
&\quad + (1 - 2^4) \frac{B_2}{13! \cdot 4} x^{13} - \dots \quad (a = 10) \\
&\vdots \\
&= \frac{(2 - 1)\pi^2}{1! \cdot 2!} B_1 x - \frac{1}{2 \cdot 3!} x^3 \quad (a = 3) \\
&= \frac{(2^3 - 1)\pi^4}{1! \cdot 4!} B_2 x - \frac{(2 - 1)\pi^2}{3! \cdot 2!} B_1 x^3 + \frac{1}{2 \cdot 5!} x^5 \quad (a = 5) \\
&= \frac{(2^5 - 1)\pi^6}{1! \cdot 6!} B_3 x - \frac{(2^3 - 1)\pi^4}{3! \cdot 4!} B_2 x^3 + \frac{(2 - 1)\pi^2}{5! \cdot 2!} B_1 x^5 - \frac{1}{2 \cdot 7!} x^7 \quad (a = 7) \\
&\vdots
\end{aligned} \tag{20}$$

$$\begin{aligned}
& \sin x + \frac{1}{3^a} \sin 3x + \frac{1}{5^a} \sin 5x + \frac{1}{7^a} \sin 7x + \frac{1}{9^a} \sin 9x \dots \\
&= \frac{\pi}{4} \quad (a = 1) \\
&= -\frac{\pi}{4 \cdot 2!} x^2 + \frac{\pi^2}{8} x \quad (a = 3) \\
&= \frac{\pi}{4 \cdot 4!} x^4 - \frac{\pi^2}{3! \cdot 8} x^3 + \frac{(2^4 - 1)\pi^4}{4! \cdot 2} B_2 x \quad (a = 5) \\
&= -\frac{\pi}{6! \cdot 4} x^6 + \frac{\pi^2}{5! \cdot 8} x^5 - \frac{(2^4 - 1)\pi^4}{3! \cdot 4 \cdot 2} B_2 x^3 + \frac{(2^6 - 1)\pi^6}{6! \cdot 2} B_3 x \quad (a = 7) \\
&\vdots
\end{aligned} \tag{21}$$

$$\begin{aligned}
& \sin x - \frac{1}{3^a} \sin 3x + \frac{1}{5^a} \sin 5x - \frac{1}{7^a} \sin 7x + \cdots \\
& = 0 \quad (a = 0) \\
& = \frac{\pi}{4} x \quad (a = 2) \\
& = \frac{\left(\frac{\pi}{2}\right)^3}{2 \cdot 2!} E_1 x - \frac{\left(\frac{\pi}{2}\right)}{2 \cdot 3! \cdot 0!} E_0 x^3 \quad (a = 4) \\
& = \frac{\left(\frac{\pi}{2}\right)^5}{2! \cdot 1! \cdot 4!} E_2 x - \frac{\left(\frac{\pi}{2}\right)^3}{2 \cdot 3! \cdot 2!} E_1 x^3 + \frac{\left(\frac{\pi}{2}\right)}{2 \cdot 5! \cdot 0!} E_0 x^5 \quad (a = 6) \\
& = \frac{\left(\frac{\pi}{2}\right)^7}{2 \cdot 1! \cdot 6!} E_3 x - \frac{\left(\frac{\pi}{2}\right)^5}{2 \cdot 3! \cdot 4!} E_2 x^3 + \frac{\left(\frac{\pi}{2}\right)^3}{2 \cdot 5! \cdot 2!} E_1 x^5 - \frac{\left(\frac{\pi}{2}\right)}{2 \cdot 7! \cdot 0!} E_0 x^7 \quad (a = 8) \\
& \vdots \\
& = \mu(0)x + \frac{E_1}{2 \cdot 3!} x^3 + \frac{E_2}{2 \cdot 5!} x^5 + \frac{E_3}{2 \cdot 7!} x^7 + \cdots \quad (a = 1) \\
& = \mu(2)x - \frac{\mu(0)}{3!} x^3 - \frac{E_1}{2 \cdot 5!} x^5 - \frac{E_2}{2 \cdot 7!} x^7 - \cdots \quad (a = 3) \\
& = \mu(4)x - \frac{\mu(2)}{3!} x^3 + \frac{\mu(0)}{5!} x^5 + \frac{E_1}{2 \cdot 7!} x^7 + \cdots \quad (a = 5) \\
& \vdots
\end{aligned} \tag{22}$$

これは $a = 0, -2, -4, -6, \dots$ で 0 となる。

$$\begin{aligned}
& \cos x + \frac{1}{2^a} \cos 2x + \frac{1}{3^a} \cos 3x + \frac{1}{4^a} \cos 4x + \frac{1}{5^a} \cos 5x + \cdots \\
& = \frac{2\pi^2}{0! \cdot 2!} B_1 - \frac{\pi}{1! \cdot 2} x + \frac{2 \cdot 2!^2}{x} \quad (a = 2) \\
& = \frac{2^3 \pi^4}{0! \cdot 4!} B_2 - \frac{2\pi^2}{2! \cdot 2!} B_1 x^2 + \frac{\pi}{3! \cdot 2} x^3 - \frac{1}{2 \cdot 4!} x^4 \quad (a = 4) \\
& = \frac{2^5 \pi^6}{0! \cdot 6!} B_3 - \frac{2^3 \pi^4}{2! \cdot 4!} B_2 x^2 + \frac{2\pi^2}{4! \cdot 2!} B_1 x^4 - \frac{\pi}{2 \cdot 5!} x^5 + \frac{1}{2 \cdot 6!} x^6 \quad (a = 6) \\
& = \frac{2^7 \pi^8}{0! \cdot 8!} B_4 - \frac{2^5 \pi^6}{2! \cdot 6!} B_3 x^2 + \frac{2^3 \pi^4}{4! \cdot 4!} B_2 x^4 - \frac{2\pi^2}{6! \cdot 2!} B_1 x^6 + \frac{\pi}{2 \cdot 7!} x^7 - \frac{1}{2 \cdot 8!} x^8 \quad (a = 8) \\
& \vdots \\
& = -\frac{1}{x^2} - \frac{B_1}{2 \cdot 0!} - \frac{B_2}{4 \cdot 2!} x^2 - \frac{B_3}{6 \cdot 4!} x^4 - \frac{B_4}{8 \cdot 6!} x^6 - \cdots \quad (a = -1) \\
& = \frac{3!}{x^4} + \frac{B_2}{4 \cdot 0!} + \frac{B_3}{6 \cdot 2!} x^2 + \frac{B_4}{8 \cdot 4!} x^4 + \frac{B_5}{10 \cdot 6!} x^6 + \cdots \quad (a = -3) \\
& = -\frac{5!}{x^6} - \frac{B_3}{6 \cdot 0!} - \frac{B_4}{8 \cdot 2!} x^2 - \frac{B_5}{10 \cdot 4!} x^4 - \frac{B_6}{12 \cdot 6!} x^6 - \cdots \quad (a = -5) \\
& \vdots
\end{aligned} \tag{23}$$

これは $a = -2, -4, -6, \dots$ で 0 、 $a = 0$ で $-\frac{1}{2}$ となる。

$$\begin{aligned}
& \cos x + \frac{1}{3^a} \cos 3x + \frac{1}{5^a} \cos 5x + \frac{1}{7^a} \cos 7x + \dots \\
&= \frac{(2^2 - 1)\pi^2}{2 \cdot 0! \cdot 2!} B_1 - \frac{\pi}{4 \cdot 1!} x \quad (a = 2) \\
&= \frac{(2^4 - 1)\pi^4}{2 \cdot 0! \cdot 4!} B_2 - \frac{(2^2 - 1)\pi^2}{2 \cdot 2! \cdot 2!} B_1 x^2 + \frac{\pi}{4 \cdot 3!} x^3 \quad (a = 4) \\
&= \frac{(2^6 - 1)\pi^6}{2 \cdot 0! \cdot 6!} B_3 - \frac{(2^4 - 1)\pi^4}{2 \cdot 2! \cdot 4!} B_2 x^2 + \frac{(2^2 - 1)\pi^2}{2 \cdot 4! \cdot 2!} B_1 x^4 - \frac{\pi}{4 \cdot 5!} \quad (a = 6) \\
&\vdots \\
&= -\frac{1}{2x^2} + \frac{B_1}{2 \cdot 0!} + \frac{(2^3 - 1)}{4 \cdot 2!} B_2 x^2 + \frac{(2^5 - 1)}{6 \cdot 4!} B_3 x^4 + \frac{(2^7 - 1)}{8 \cdot 6!} B_4 x^6 + \dots \quad (a = -1) \\
&= \frac{3!}{2x^4} - \frac{(2^3 - 1)}{4 \cdot 0!} B_2 - \frac{(2^5 - 1)}{6 \cdot 2!} B_3 x^2 - \frac{(2^7 - 1)}{8 \cdot 4!} B_4 x^4 - \frac{(2^9 - 1)}{10 \cdot 6!} B_5 x^6 - \dots \quad (a = -3) \\
&= -\frac{5!}{2x^6} + \frac{(2^5 - 1)}{6 \cdot 0!} B_3 + \frac{(2^7 - 1)}{8 \cdot 2!} B_4 x^2 + \frac{(2^9 - 1)}{10 \cdot 4!} B_5 x^4 + \frac{(2^{11} - 1)}{12 \cdot 6!} B_6 x^6 + \dots \quad (a = -5) \\
&\vdots
\end{aligned} \tag{24}$$

これは $a = 0, -2, -4, -6, \dots$ で 0 となる。

$$\begin{aligned}
& \cos x - \frac{1}{2^a} \cos 2x + \frac{1}{3^a} \cos 3x - \frac{1}{4^a} \cos 4x + \frac{1}{5^a} \cos 5x - \dots \\
&= \frac{\pi^2}{0! \cdot 2!} B_1 - \frac{1}{2 \cdot 2!} x^2 \quad (a = 2) \\
&= \frac{(2^3 - 1)\pi^4}{0! \cdot 4!} B_2 - \frac{\pi^2}{2! \cdot 2!} B_1 x^2 + \frac{1}{2 \cdot 4!} x^4 \quad (a = 4) \\
&= \frac{(2^5 - 1)\pi^6}{0! \cdot 6!} B_3 - \frac{(2^3 - 1)\pi^4}{2! \cdot 4!} B_2 x^2 + \frac{\pi^2}{4! \cdot 2!} B_1 x^4 - \frac{1}{2 \cdot 6!} x^6 \quad (a = 6) \\
&\vdots \\
&= \ln 2 - \frac{2^2 - 1}{2 \cdot 2!} B_1 x^2 - \frac{2^4 - 1}{4 \cdot 4!} B_2 x^4 - \frac{2^6 - 1}{6 \cdot 6!} B_3 x^6 - \dots \quad (a = 1) \\
&= \frac{2^2 - 1}{2^2} \zeta(3) - \frac{\ln 2}{2!} x^2 + \frac{2^2 - 1}{2 \cdot 4!} B_1 x^4 + \frac{2^4 - 1}{4 \cdot 6!} B_2 x^6 + \dots \quad (a = 3) \\
&= \frac{2^4 - 1}{2^4} \zeta(5) - \frac{2^2 - 1}{2^2 \cdot 2!} \zeta(3) x^2 + \frac{\ln 2}{4!} x^4 - \frac{2^2 - 1}{2 \cdot 6!} B_1 x^6 - \dots \quad (a = 5) \\
&= \frac{2^6 - 1}{2^6} \zeta(7) - \frac{2^4 - 1}{2^4 \cdot 2!} \zeta(5) x^2 + \frac{2^2 - 1}{2^2 \cdot 4!} \zeta(3) x^4 - \frac{\ln 2}{6!} x^6 + \dots \quad (a = 7) \\
&\vdots \\
&= \frac{2^2 - 1}{2 \cdot 0!} B_1 + \frac{2^4 - 1}{4 \cdot 2!} B_2 x^2 + \frac{2^6 - 1}{6 \cdot 4!} B_3 x^4 + \frac{2^8 - 1}{8 \cdot 6!} B_4 x^6 + \dots \quad (a = -1) \\
&= -\frac{2^4 - 1}{4 \cdot 0!} B_2 - \frac{2^6 - 1}{6 \cdot 2!} B_3 x^2 - \frac{2^8 - 1}{8 \cdot 4!} B_4 x^4 - \frac{2^{10} - 1}{10 \cdot 6!} B_5 x^6 - \dots \quad (a = -3) \\
&= \frac{2^6 - 1}{6 \cdot 0!} B_3 + \frac{2^8 - 1}{8 \cdot 2!} B_4 x^2 + \frac{2^{10} - 1}{10 \cdot 4!} x^4 + \frac{2^{12} - 1}{12 \cdot 6!} B_6 x^6 + \dots \quad (a = -5) \\
&\vdots
\end{aligned} \tag{25}$$

これは $a = -2, -4, -6, \dots$ で $= 0$ となり、 $a = 0$ で $= \frac{1}{2}$ となる。

$$\begin{aligned}
& \cos x - \frac{1}{3^a} \cos 3x + \frac{1}{5^a} \cos 5x - \frac{1}{7^a} \cos 7x + \frac{1}{9^a} \cos 9x - \dots \\
&= \frac{\pi}{4} \quad (a = 1) \\
&= \frac{\pi^3}{2^4 \cdot 2!} E_1 - \frac{\pi}{4 \cdot 2!} x^2 \quad (a = 3) \\
&= \frac{\pi^5}{2^6 \cdot 4!} E_2 - \frac{\pi^3}{2^4 \cdot 2! \cdot 2!} E_1 x^2 + \frac{\pi}{4 \cdot 4!} x^4 \quad (a = 5) \\
&= \frac{\pi^7}{2^8 \cdot 6!} E_3 - \frac{\pi^5}{2^6 \cdot 4! \cdot 2!} E_2 x^2 + \frac{\pi^3}{2^4 \cdot 2! \cdot 4!} E_1 x^4 - \frac{\pi}{4 \cdot 6!} x^6 \quad (a = 7) \\
&\vdots \\
&= \mu(2) - \frac{1}{2 \cdot 2!} x^2 - \frac{1}{2 \cdot 2!} x^2 - \frac{E_1}{2 \cdot 4!} x^4 - \frac{E_2}{2 \cdot 6!} x^6 - \frac{E_3}{2 \cdot 8!} x^8 - \dots \quad (a = 2) \\
&= \mu(4) - \frac{1}{2!} \mu(2) x^2 + \frac{1}{2 \cdot 4!} x^4 + \frac{E_1}{2 \cdot 6!} x^6 + \frac{E_2}{2 \cdot 8!} x^8 + \dots \quad (a = 4) \\
&= \mu(6) - \frac{1}{2!} \mu(4) x^2 + \frac{1}{4!} \mu(2) x^4 - \frac{1}{2 \cdot 6!} x^6 - \frac{E_1}{2 \cdot 8!} x^8 - \dots \quad (a = 6) \\
&\vdots \\
&= \frac{1}{2} + \frac{E_1}{2 \cdot 2!} x^2 + \frac{E_2}{2 \cdot 4!} x^4 + \frac{E_3}{2 \cdot 6!} x^6 + \frac{E_4}{2 \cdot 8!} x^8 + \dots \quad (a = 0) \\
&= -\frac{E_1}{2} - \frac{E_2}{2 \cdot 2!} x^2 - \frac{E_3}{2 \cdot 4!} x^4 - \frac{E_4}{2 \cdot 6!} x^6 - \frac{E_5}{2 \cdot 8!} x^8 - \dots \quad (a = -2) \\
&= \frac{E_1}{2} + \frac{E_3}{2 \cdot 2!} x^2 + \frac{E_4}{2 \cdot 4!} x^4 + \frac{E_5}{2 \cdot 6!} x^6 + \frac{E_6}{2 \cdot 8!} x^8 + \dots \quad (a = -4) \\
&\vdots
\end{aligned} \tag{26}$$

5 ディリクレ級数の1例

上で求めた公式をさらに利用して

$$\begin{aligned}
\frac{\sin x}{1-u^2} + \frac{2 \sin 2x}{2^2-u^2} + \frac{3 \sin 3x}{3^2-u^2} + \frac{4 \sin 4x}{4^2-u^2} + \frac{5 \sin 5x}{5^2-u^2} + \dots &= \frac{\pi}{2} \cos xu - \frac{\pi}{2} \cot \pi u \cdot \sin xu \\
&= \frac{\pi \sin\{(\pi-x)u\}}{2 \sin \pi u}
\end{aligned} \tag{27}$$

これは

$$\sum_{n=1}^{\infty} \frac{n \sin nx}{n^2 - u^2} = \sum_{m=0}^{\infty} u^{2m} \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m+1}} \tag{28}$$

を (19) を用いて u と x の冪級数で表し、それらから係数を比較し右辺を求める。同様の計算で

$$\frac{\sin x}{1-u^2} - \frac{2 \sin 2x}{2^2-u^2} + \frac{3 \sin 3x}{3^2-u^2} - \frac{4 \sin 4x}{4^2-u^2} + \frac{5 \sin 5x}{5^2-u^2} - \dots = \frac{\pi \sin xu}{2 \sin \pi u} \tag{29}$$

$$\begin{aligned}
\frac{\sin x}{1-u^2} + \frac{3 \sin 3x}{3^2-u^2} + \frac{5 \sin 5x}{5^2-u^2} + \frac{7 \sin 7x}{7^2-u^2} + \frac{9 \sin 9x}{9^2-u^2} + \dots &= \frac{\pi}{4} \cos xu + \frac{\pi}{4} \tan \frac{\pi}{2} u \cdot \sin ux \\
&= \frac{\pi \cos\{(x - \frac{\pi}{2})u\}}{4 \cos \frac{\pi}{2} u}
\end{aligned} \tag{30}$$

$$\frac{\sin x}{1-u^2} - \frac{\sin 3x}{3^2-u^2} + \frac{\sin 5x}{5^2-u^2} - \frac{\sin 7x}{7^2-u^2} + \frac{\sin 9x}{9^2-u^2} - \cdots = \frac{\pi \sin xu}{4u \cos \frac{\pi}{2}u} \quad (31)$$

$$\begin{aligned} \frac{\cos x}{1-u^2} + \frac{2^2 \cos 2x}{2^2-u^2} + \frac{3^2 \cos 3x}{3^2-u^2} + \frac{4^2 \cos 4x}{4^2-u^2} + \frac{5^2 \cos 5x}{5^2-u^2} + \cdots &= -\frac{\pi u}{2} \left(\sin xu + \cot \pi u \cdot \cos xu \right) \\ &= -\frac{\pi u \cos \{(x-\pi)u\}}{2 \sin \pi u} \end{aligned} \quad (32)$$

$$\frac{\cos x}{1-u^2} - \frac{2^2 \cos 2x}{2^2-u^2} + \frac{3^2 \cos 3x}{3^2-u^2} - \frac{4^2 \cos 4x}{4^2-u^2} + \frac{5^2 \cos 5x}{5^2-u^2} - \cdots = \frac{\pi u \cos xu}{2 \sin \pi u} \quad (33)$$

$$\begin{aligned} \frac{\cos x}{1-u^2} + \frac{\cos 3x}{3^2-u^2} + \frac{\cos 5x}{5^2-u^2} + \frac{\cos 7x}{7^2-u^2} + \frac{\cos 9x}{9^2-u^2} + \cdots &= \frac{\pi}{4u} \left(\tan \frac{\pi u}{2} \cdot \cos xu - \sin xu \right) \\ &= \frac{\pi \sin \{(\frac{\pi}{2}-x)u\}}{4u \cos \frac{\pi}{2}u} \end{aligned} \quad (34)$$

$$\frac{\cos x}{1-u^2} - \frac{3 \cos 3x}{3^2-u^2} + \frac{5 \cos 5x}{5^2-u^2} - \frac{7 \cos 7x}{7^2-u^2} + \frac{9 \cos 9x}{9^2-u^2} - \cdots = \frac{\pi \cos xu}{4 \cos \frac{\pi}{2}u} \quad (35)$$

次にゼータ関数の積分表示を応用して級数の積分表示を求める。

$$\int_0^\infty \frac{x^{s-1}}{e^{bx}(e^{ax}-1)} dx = \Gamma(s) \left(\frac{1}{(b+a)^s} + \frac{1}{(b+2a)^s} + \frac{1}{(b+3a)^s} + \cdots \right) \quad (36)$$

$$\int_0^\infty \frac{e^{bx} x^{s-1}}{e^{ax} - e^{-ax}} dx = \Gamma(s) \left(\frac{1}{(a-b)^s} + \frac{1}{(3a-b)^s} + \frac{1}{(5a-b)^s} + \cdots \right) \quad (37)$$

$$\begin{aligned} \int_0^\infty \frac{\cosh bx}{\sinh ax} x^{s-1} dx &= \int_0^\infty \frac{e^{bx} + e^{-bx}}{e^{ax} - e^{-ax}} x^{s-1} dx \\ &= \Gamma(s) \left(\frac{1}{(a-b)^s} + \frac{1}{(3a-b)^s} + \frac{1}{(5a-b)^s} + \cdots \right. \\ &\quad \left. + \frac{1}{(a+b)^s} + \frac{1}{(3a+b)^s} + \frac{1}{(5a+b)^s} + \cdots \right) \end{aligned} \quad (38)$$

$$\begin{aligned} \int_0^\infty \frac{\sinh bx}{\sinh ax} x^{s-1} dx &= \Gamma(s) \left(\frac{1}{(a-b)^s} + \frac{1}{(3a-b)^s} + \frac{1}{(5a-b)^s} + \cdots \right. \\ &\quad \left. - \frac{1}{(a+b)^s} - \frac{1}{(3a+b)^s} - \frac{1}{(5a+b)^s} - \cdots \right) \end{aligned} \quad (39)$$

などを得る。

次に上で求めた公式からゼータ関数の形の級数の積分表示を求める。例えば

$$\begin{aligned} \int_0^\infty u^{s-1} \frac{\sinh \{(\pi-x)u\}}{\sinh \pi u} du &= \frac{2}{\pi} \int_0^\infty u^{s-1} \sum_{n=1}^\infty \frac{n \sin nx}{n^2 + u^2} du \\ &= \frac{2}{\pi} \sum_{n=1}^\infty n \sin nx \int_0^\infty \frac{u^{s-1}}{n^2 + u^2} du \\ &= \frac{1}{\sin \frac{\pi}{2}s} \sum_{n=1}^\infty \frac{\sin nx}{n^{1-s}} \end{aligned} \quad (40)$$

として求められる。同様の計算をすることで似た形の他のゼータ関数の形の級数の積分表示を求められる。まとめると

$$\phi_1(s) = \frac{\sin x}{1^s} + \frac{\sin 2x}{2^s} + \frac{\sin 3x}{3^s} + \frac{\sin 4x}{4^s} + \dots \quad (41)$$

$$\phi_2(s) = \frac{\sin x}{1^s} - \frac{\sin 2x}{2^s} + \frac{\sin 3x}{3^s} - \frac{\sin 4x}{4^s} + \dots \quad (42)$$

$$\phi_3(s) = \frac{\sin x}{1^s} + \frac{\sin 3x}{3^s} + \frac{\sin 5x}{5^s} + \frac{\sin 7x}{7^s} + \dots \quad (43)$$

$$\phi_4(s) = \frac{\sin x}{1^s} - \frac{\sin 3x}{3^s} + \frac{\sin 5x}{5^s} - \frac{\sin 7x}{7^s} + \dots \quad (44)$$

and

$$\psi_1(s) = \frac{\cos x}{1^s} + \frac{\cos 2x}{2^s} + \frac{\cos 3x}{3^s} + \frac{\cos 4x}{4^s} + \dots \quad (45)$$

$$\psi_2(s) = \frac{\cos x}{1^s} - \frac{\cos 2x}{2^s} + \frac{\cos 3x}{3^s} - \frac{\cos 4x}{4^s} + \dots \quad (46)$$

$$\psi_3(s) = \frac{\cos x}{1^s} + \frac{\cos 3x}{3^s} + \frac{\cos 5x}{5^s} + \frac{\cos 7x}{7^s} + \dots \quad (47)$$

$$\psi_4(s) = \frac{\cos x}{1^s} - \frac{\cos 3x}{3^s} + \frac{\cos 5x}{5^s} - \frac{\cos 7x}{7^s} + \dots \quad (48)$$

(49)

と置いた時

$$\phi_1(1-s) = \sin \frac{\pi}{2}s \int_0^\infty u^{s-1} \frac{\sinh\{(\pi-x)u\}}{\sinh \pi u} du \quad (50)$$

$$\phi_2(1-s) = \sin \frac{\pi}{2}s \int_0^\infty u^{s-1} \frac{\sinh xu}{\sinh \pi u} du \quad (51)$$

$$\phi_3(1-s) = \frac{\sin \frac{\pi}{2}s}{2} \int_0^\infty u^{s-1} \frac{\cosh\{(x-\frac{\pi}{2})u\}}{\cosh \frac{\pi}{2}u} du \quad (52)$$

$$\phi_4(1-s) = \frac{\cos \frac{\pi}{2}s}{2} \int_0^\infty u^{s-1} \frac{\sinh xu}{\cosh \frac{\pi}{2}u} du \quad (53)$$

and

$$\psi_1(1-s) = \cos \frac{\pi}{2}s \int_0^\infty u^{s-1} \frac{\cosh\{(x-\pi)u\}}{\sinh \pi u} du \quad (54)$$

$$\psi_2(1-s) = -\cos \frac{\pi}{2}s \int_0^\infty u^{s-1} \frac{\cosh xu}{\sinh \pi u} du \quad (55)$$

$$\psi_3(1-s) = \frac{\cos \frac{\pi}{2}s}{2} \int_0^\infty u^{s-1} \frac{\sinh\{(\frac{\pi}{2}-x)u\}}{\cosh \frac{\pi}{2}u} du \quad (56)$$

$$\psi_4(1-s) = \frac{\sin \frac{\pi}{2}s}{2} \int_0^\infty u^{s-1} \frac{\cosh xu}{\cosh \frac{\pi}{2}u} du \quad (57)$$

次に

$$\begin{aligned} \psi_1(s) + i\phi_1(s) &= \sum_{n=1}^{\infty} \frac{e^{inx}}{n^s} \\ &= \sum_{n=1}^{\infty} e^{inx} \frac{1}{\Gamma(s)} \int_0^\infty e^{-nu} u^{s-1} du \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{ix-u}}{1-e^{ix-u}} u^{s-1} du \\ &= \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{e^{ix} - e^{-u}}{e^u - 2\cos x + e^{-u}} du \end{aligned} \quad (58)$$

となることより、実部と虚部を比較すれば $\phi_1(s)$ と $\psi_1(s)$ の積分表示も得られる。他も同様である。まとめて書くと

$$\phi_1(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\sin x}{e^u - 2 \cos x + e^{-u}} du \quad (59)$$

$$\phi_2(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\sin x}{e^u + 2 \cos x + e^{-u}} du \quad (60)$$

$$\phi_3(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\sin x (e^u + e^{-u})}{e^{2u} - 2 \cos 2x + e^{-2u}} du \quad (61)$$

$$\phi_4(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\sin x (e^u - e^{-u})}{e^{2u} + 2 \cos 2x + e^{-2u}} du \quad (62)$$

and

$$\psi_1(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\cosh(x - \pi)u}{\sinh \pi u} du \quad (63)$$

$$\psi_2(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\cos x + e^{-u}}{e^u + 2 \cos x + e^{-u}} du \quad (64)$$

$$\psi_3(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\cos x (e^u - e^{-u})}{e^{2u} - 2 \cos 2x + e^{-2u}} du \quad (65)$$

$$\psi_4(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{\cos x (e^u + e^{-u})}{e^{2u} + 2 \cos 2x + e^{-2u}} du \quad (66)$$

となる。ここで (50) から (57) 式までに対して、(36) から (39) などを用いると、

$$\phi_1(1-s) = \Gamma(s) \sin \frac{\pi}{2} s \left(\frac{1}{x^s} + \frac{1}{(2\pi+x)^s} - \frac{1}{(2\pi-x)^s} + \frac{1}{(4\pi+x)^s} - \frac{1}{(4\pi-x)^s} + \dots \right) \quad (67)$$

$$\phi_2(1-s) = \Gamma(s) \sin \frac{\pi}{2} s \left(\frac{1}{(\pi-x)^s} - \frac{1}{(\pi+x)^s} + \frac{1}{(3\pi-x)^s} - \frac{1}{(3\pi+x)^s} + \frac{1}{(5\pi-x)^s} - \frac{1}{(5\pi+x)^s} + \dots \right) \quad (68)$$

$$\phi_3(1-s) = \frac{1}{2} \Gamma(s) \sin \frac{\pi}{2} s \left(\frac{1}{x^s} - \frac{1}{(\pi+x)^s} + \frac{1}{(\pi-x)^s} + \frac{1}{(2\pi+x)^s} - \frac{1}{(2\pi-x)^s} + \dots \right) \quad (69)$$

$$\phi_4(1-s) = \frac{1}{2} \Gamma(s) \cos \frac{\pi}{2} s \left(\frac{1}{(\frac{\pi}{2}-x)^s} - \frac{1}{(\frac{\pi}{2}+x)^s} - \frac{1}{(\frac{3}{2}\pi-x)^s} + \frac{1}{(\frac{3}{2}\pi+x)^s} + \frac{1}{(\frac{5}{2}\pi-x)^s} - \frac{1}{(\frac{5}{2}\pi+x)^s} + \dots \right) \quad (70)$$

and

$$\psi_1(1-s) = \Gamma(s) \cos \frac{\pi}{2} s \left(\frac{1}{x^s} + \frac{1}{(2\pi-x)^s} + \frac{1}{(2\pi+x)^s} + \frac{1}{(4\pi-x)^s} + \frac{1}{(4\pi+x)^s} + \dots \right) \quad (71)$$

$$\psi_2(1-s) = -\Gamma(s) \cos \frac{\pi}{2} s \left(\frac{1}{(\pi-x)^s} + \frac{1}{(\pi+x)^s} + \frac{1}{(3\pi-x)^s} + \frac{1}{(3\pi+x)^s} + \frac{1}{(5\pi-x)^s} + \frac{1}{(5\pi+x)^s} + \dots \right) \quad (72)$$

$$\psi_3(1-s) = \frac{1}{2} \Gamma(s) \cos \frac{\pi}{2} s \left(\frac{1}{x^s} - \frac{1}{(\pi+x)^s} - \frac{1}{(\pi-x)^s} + \frac{1}{(2\pi+x)^s} + \frac{1}{(2\pi-x)^s} + \dots \right) \quad (73)$$

$$\psi_4(1-s) = \frac{1}{2} \Gamma(s) \sin \frac{\pi}{2} s \left(\frac{1}{(\frac{\pi}{2}+x)^s} + \frac{1}{(\frac{\pi}{2}-x)^s} - \frac{1}{(\frac{3}{2}\pi+x)^s} - \frac{1}{(\frac{3}{2}\pi-x)^s} + \frac{1}{(\frac{5}{2}\pi+x)^s} + \frac{1}{(\frac{5}{2}\pi-x)^s} + \dots \right) \quad (74)$$

を得る。

6 いろいろな級数を多項式やベキ級数で表現する

ここまでのことを踏まえていろいろな級数を多項式で表現していく。

$$\begin{aligned}
& \frac{1}{e^x - 1} + \frac{2^a}{e^{2x} - 1} + \frac{3^a}{e^{3x} - 1} + \frac{4^a}{e^{4x} - 1} + \frac{5^a}{e^{5x} - 1} + \dots \\
&= \frac{(2\pi)^2}{2 \cdot 2} B_1 \frac{1}{x^2} - \frac{1}{2x} + \frac{1}{2 \cdot 2} B_1 \quad (a=1) \\
&= \frac{(2\pi)^4}{2 \cdot 4} B_2 \frac{1}{x^4} - \frac{1}{2 \cdot 4} B_2 \quad (a=3) \\
&= \frac{(2\pi)^6}{2 \cdot 6} B_3 \frac{1}{x^6} + \frac{1}{2 \cdot 6} B_3 \quad (a=5) \\
&= \frac{(2\pi)^8}{2 \cdot 8} B_4 \frac{1}{x^8} - \frac{1}{2 \cdot 8} B_4 \quad (a=7) \\
&\vdots \\
&= 2! \cdot \zeta(3) \frac{1}{x^3} - \frac{B_1}{2x} + \frac{B_1 B_2}{1! \cdot 2 \cdot 4} x + \frac{B_2 B_3}{3! \cdot 4 \cdot 6} x^3 + \frac{B_3 B_4}{5! \cdot 6 \cdot 8} x^5 + \dots \quad (a=2) \\
&= 4! \cdot \zeta(5) \frac{1}{x^5} + \frac{B_2}{4x} - \frac{B_1 B_3}{1! \cdot 2 \cdot 6} x - \frac{B_2 B_4}{3! \cdot 4 \cdot 8} x^3 - \frac{B_3 B_5}{5! \cdot 6 \cdot 10} x^5 - \dots \quad (a=4) \\
&= 6! \cdot \zeta(7) \frac{1}{x^7} - \frac{B_3}{6x} + \frac{B_1 B_4}{1! \cdot 2 \cdot 8} x + \frac{B_2 B_5}{3! \cdot 4 \cdot 10} x^3 + \frac{B_3 B_6}{5! \cdot 6 \cdot 12} x^5 + \dots \quad (a=6) \\
&\vdots \\
&= \frac{(2\pi)^4}{2 \cdot 4!} B_2 \frac{1}{x} - \frac{1}{2} \zeta(3) + \frac{(2\pi)^2 B_1^2}{2 \cdot 2 \cdot 2! \cdot 1!} x + \frac{B_2}{2 \cdot 4 \cdot 3!} x^3 - \frac{\zeta(3)}{2 \cdot (2\pi)^2} x^2 \quad (a=-3) \\
&= \frac{(2\pi)^6}{2 \cdot 6!} B_3 \frac{1}{x} - \frac{1}{2} \zeta(5) + \frac{(2\pi)^4 B_1 B_2}{2 \cdot 2 \cdot 4! \cdot 1!} x - \frac{(2\pi)^2 B_1 B_2}{2 \cdot 4 \cdot 2! \cdot 3!} x^3 - \frac{B_3}{2 \cdot 6 \cdot 5!} x^5 + \frac{\zeta(5)}{2 \cdot (2\pi)^4} x^4 \quad (a=-5) \\
&= \frac{(2\pi)^8}{2 \cdot 8!} B_4 \frac{1}{x} - \frac{1}{2} \zeta(7) + \frac{(2\pi)^6 B_1 B_3}{2 \cdot 2 \cdot 6! \cdot 1!} x - \frac{(2\pi)^4 B_2^2}{2 \cdot 4 \cdot 3! \cdot 4!} x^3 + \frac{(2\pi)^2 B_1 B_3}{2 \cdot 6 \cdot 2! \cdot 5!} x^5 + \frac{B_4}{2 \cdot 8 \cdot 7!} x^7 - \frac{\zeta(7)}{2 \cdot (2\pi)^6} x^6 \quad (a=-7) \\
&\vdots
\end{aligned} \tag{75}$$

$$\begin{aligned}
& \frac{1}{e^x - 1} - \frac{2^a}{e^{2x} - 1} + \frac{3^a}{e^{3x} - 1} - \frac{4^a}{e^{4x} - 1} + \frac{5^a}{e^{5x} - 1} - \dots \\
&= \frac{1}{2x} - \frac{2^2 - 1}{2 \cdot 2} B_1 \quad (a = 1) \\
&= \frac{2^4 - 1}{2 \cdot 4} B_2 \quad (a = 3) \\
&= -\frac{2^6 - 1}{2 \cdot 6} B_3 \quad (a = 5) \\
&= \frac{2^8 - 1}{2 \cdot 8} B_4 \quad (a = 7) \\
&\vdots \\
&= \ln 2 \cdot \frac{1}{x} - \frac{1}{4} + \frac{2^2 - 1}{2^2} B_1^2 x + \frac{2^4 - 1}{3! \cdot 4^2} B_2^2 x^3 + \frac{2^6 - 1}{5! \cdot 6^2} B_3^2 x^5 + \dots \quad (a = 0) \\
&= \frac{2^2 - 1}{2} B_1 \frac{1}{x} - \frac{(2^4 - 1) B_1 B_2}{2 \cdot 4 \cdot 1!} x - \frac{(2^6 - 1) B_2 B_3}{4 \cdot 6 \cdot 3!} x^3 - \frac{(2^8 - 1) B_3 B_4}{6 \cdot 8 \cdot 5!} x^5 - \dots \quad (a = 2) \\
&= -\frac{2^4 - 1}{4} B_2 \frac{1}{x} + \frac{(2^6 - 1) B_1 B_3}{2 \cdot 6 \cdot 1!} x + \frac{(2^8 - 1) B_2 B_4}{4 \cdot 8 \cdot 3!} x^3 + \frac{(2^{10} - 1) B_3 B_5}{6 \cdot 10 \cdot 5!} x^5 + \dots \quad (a = 4) \\
&= \frac{2^6 - 1}{6} B_3 \frac{1}{x} - \frac{(2^8 - 1) B_1 B_4}{2 \cdot 8 \cdot 1!} x - \frac{(2^{10} - 1) B_2 B_5}{4 \cdot 10 \cdot 3!} x^3 - \frac{(2^{12} - 1) B_3 B_6}{6 \cdot 12 \cdot 5!} x^5 - \dots \quad (a = 6) \\
&\vdots \\
&= \frac{(2 - 1) \pi^2}{2!} B_1 \frac{1}{x} - \frac{\ln 2}{2} + \frac{(2 - 1) B_1}{2 \cdot 2 \cdot 1!} x \quad (a = -1) \\
&= \frac{(2^3 - 1) \pi^4}{4!} B_2 \frac{1}{x} - \frac{2^2 - 1}{2^3} \zeta(3) + \frac{(2 - 1) \pi^2 B_1 B_1}{2 \cdot 1! \cdot 2!} x - \frac{B_2}{2 \cdot 4 \cdot 3!} x^3 \quad (a = -3) \\
&= \frac{(2^5 - 1) \pi^6}{6!} B_3 \frac{1}{x} - \frac{2^4 - 1}{2^5} \zeta(5) + \frac{(2^3 - 1) \pi^4 B_1 B_2}{2 \cdot 1! \cdot 4!} x - \frac{\pi^2 B_2 B_1}{4 \cdot 3! \cdot 2!} x^3 + \frac{B_3}{2 \cdot 6 \cdot 5!} x^5 \quad (a = -5) \\
&= \frac{(2^7 - 1) \pi^8}{8!} B_4 \frac{1}{x} - \frac{2^6 - 1}{2^7} \zeta(7) + \frac{(2^5 - 1) \pi^6 B_1 B_3}{2 \cdot 1! \cdot 6!} x - \frac{(2^3 - 1) \pi^4 B_2 B_2}{4 \cdot 3! \cdot 4!} x^3 + \frac{\pi^2 B_3 B_1}{6 \cdot 2! \cdot 5!} x^5 - \frac{B_4}{2 \cdot 8 \cdot 7!} x^7 \quad (a = -7) \\
&\vdots \\
&= \frac{2^7 - 1}{2^2} \zeta(3) \frac{1}{x} - \frac{\pi^2 B_1}{2 \cdot 2!} + \frac{\ln 2 \cdot B_1}{2 \cdot 1!} x - \frac{(2^2 - 1) B_1 B_2}{2 \cdot 4 \cdot 3!} x^3 - \frac{(2^4 - 1) B_2 B_3}{4 \cdot 6 \cdot 5!} x^5 - \dots \quad (a = -2) \\
&= \frac{2^4 - 1}{2^4} \zeta(5) \frac{1}{x} - \frac{(2^3 - 1) \pi^4 B_2}{2 \cdot 4!} + \frac{(2^2 - 1) B_1}{2 \cdot 2^2} \zeta(3) x - \frac{\ln 2 \cdot B_2}{4 \cdot 3!} x^3 + \frac{(2^2 - 1) B_1 B_3}{2 \cdot 6 \cdot 5!} x^5 + \frac{(2^4 - 1) B_2 B_4}{4 \cdot 8 \cdot 7!} x^7 \dots \quad (a = -4) \\
&= \frac{2^6 - 1}{2^6} \zeta(7) \frac{1}{x} - \frac{(2^5 - 1) \pi^6 B_3}{2 \cdot 6!} + \frac{(2^4 - 1) B_1}{2 \cdot 2^4} \zeta(5) x - \frac{(2^2 - 1) B_2}{4 \cdot 2^2 \cdot 3!} \zeta(3) x^3 + \frac{\ln 2 \cdot B_3}{6 \cdot 5!} x^5 - \frac{(2^2 - 1) B_1 B_4}{2 \cdot 8 \cdot 7!} x^7 \\
&\quad - \frac{(2^4 - 1) B_2 B_5}{4 \cdot 10 \cdot 9!} x^9 - \dots \quad (a = -6) \\
&\vdots
\end{aligned} \tag{76}$$

$$\begin{aligned}
& \frac{1}{e^x - 1} + \frac{3^a}{e^{3x} - 1} + \frac{5^a}{e^{5x} - 1} + \frac{7^a}{e^{7x} - 1} + \frac{9^a}{e^{9x} - 1} + \cdots \\
&= \frac{1}{2} \frac{(2\pi)^2}{2 \cdot 2} B_1 \frac{1}{x^2} - \frac{B_1}{2 \cdot 2} \quad (a = 1) \\
&= \frac{1}{2} \frac{(2\pi)^4}{2 \cdot 4} B_2 \frac{1}{x^4} + \frac{2^3 - 1}{2} \frac{B_2}{4} \quad (a = 3) \\
&= \frac{1}{2} \frac{(2\pi)^6}{2 \cdot 6} B_3 \frac{1}{x^6} - \frac{2^5 - 1}{2} \frac{B_3}{6} \quad (a = 5) \\
&= \frac{1}{2} \frac{(2\pi)^8}{2 \cdot 8} B_4 \frac{1}{x^8} + \frac{2^7 - 1}{2} \frac{B_4}{8} \quad (a = 7) \\
&\vdots \\
&= \frac{2!}{2} \zeta(3) \frac{1}{x^3} + \frac{B_1}{2} \frac{1}{x} - \frac{(2^3 - 1)B_1 B_2}{2 \cdot 4 \cdot 1!} x - \frac{(2^5 - 1)B_2 B_3}{4 \cdot 6 \cdot 3!} x^3 - \cdots \quad (a = 2) \\
&= \frac{4!}{2} \zeta(5) \frac{1}{x^5} - \frac{2^3 - 1}{4} B_2 \frac{1}{x} + \frac{(2^5 - 1)B_1 B_3}{2 \cdot 6 \cdot 1!} x + \frac{(2^7 - 1)B_2 B_4}{4 \cdot 8 \cdot 3!} x^3 + \cdots \quad (a = 4) \\
&= \frac{6!}{2} \zeta(7) \frac{1}{x^7} + \frac{2^5 - 1}{6} B_3 \frac{1}{x} - \frac{(2^7 - 1)B_1 B_4}{2 \cdot 8 \cdot 1!} x - \frac{(2^9 - 1)B_2 B_5}{2 \cdot 10 \cdot 3!} x^3 - \cdots \quad (a = 6) \\
&\vdots \\
&= \frac{(2^4 - 1)\pi^4}{2 \cdot 4!} B_2 \frac{1}{x} - \frac{2^3 - 1}{2^4} \zeta(3) + \frac{(2\pi)^2 B_1 B_1}{2 \cdot 2 \cdot 1! \cdot 2!} \frac{2^2 - 1}{2^2} x - \frac{1}{2^4 \pi^2} x^2 \quad (a = -3) \\
&= \frac{(2^6 - 1)\pi^6}{2 \cdot 6!} B_3 \frac{1}{x} - \frac{2^5 - 1}{2^6} \zeta(5) + \frac{(2\pi)^4 B_1 B_2}{2 \cdot 2 \cdot 1! \cdot 4!} \frac{2^4 - 1}{2^4} x - \frac{(2\pi)^2 B_2 B_1}{2 \cdot 4 \cdot 3! \cdot 2!} \frac{2^2 - 1}{2^2} x^3 + \frac{1}{2^6 \pi^4} x^4 \quad (a = -5) \\
&= \frac{(2^8 - 1)\pi^8}{2 \cdot 8!} B_4 \frac{1}{x} - \frac{2^7 - 1}{2^8} \zeta(7) + \frac{(2\pi)^6 B_1 B_3}{2 \cdot 2 \cdot 1! \cdot 6!} \frac{2^6 - 1}{2^6} x - \frac{(2\pi)^4 B_2 B_2}{2 \cdot 4 \cdot 3! \cdot 4!} \frac{2^4 - 1}{2^4} x^3 + \frac{(2\pi)^2 B_3 B_1}{2 \cdot 6 \cdot 5! \cdot 2!} \frac{2^2 - 1}{2^2} x^5 - \frac{1}{2^8 \pi^6} x^6 \quad (a = -7) \\
&\vdots
\end{aligned} \tag{77}$$

$$\begin{aligned}
& \frac{1}{e^x - 1} - \frac{3^a}{e^{3x} - 1} + \frac{5^a}{e^{5x} - 1} - \frac{7^a}{e^{7x} - 1} + \frac{9^a}{e^{9x} - 1} - \dots \\
&= \frac{E_0}{2} \frac{1}{x} - \frac{B_1 E_1}{2 \cdot 2 \cdot 1!} x - \frac{B_2 E_2}{2 \cdot 4 \cdot 3!} x^3 - \frac{B_3 E_3}{2 \cdot 6 \cdot 5!} x^5 - \dots \quad (a = 1) \\
&= -\frac{E_1}{2} \frac{1}{x} + \frac{B_1 E_2}{2 \cdot 2 \cdot 1!} x + \frac{B_2 E_3}{2 \cdot 4 \cdot 3!} x^3 + \frac{B_3 E_4}{2 \cdot 6 \cdot 5!} x^5 + \dots \quad (a = 3) \\
&= \frac{E_2}{2} \frac{1}{x} - \frac{B_1 E_3}{2 \cdot 2 \cdot 1!} x - \frac{B_2 E_4}{2 \cdot 4 \cdot 3!} x^3 - \frac{B_3 E_5}{2 \cdot 6 \cdot 5!} x^5 - \dots \quad (a = 5) \\
&\vdots \\
&= \frac{\pi}{4} \frac{1}{x} - \frac{1}{4} \quad (a = 0) \\
&= \frac{E_1}{4} \quad (a = 2) \\
&= -\frac{E_2}{4} \quad (a = 4) \\
&= \frac{E_3}{4} \quad (a = 6) \\
&\vdots \\
&= \mu(2) \frac{1}{x} - \frac{\pi}{8} + \frac{B_1 E_0}{2 \cdot 2 \cdot 1!} x + \frac{B_2 E_1}{2 \cdot 4 \cdot 3!} x^3 + \frac{B_3 E_2}{2 \cdot 6 \cdot 5!} x^5 + \dots \quad (a = -1) \\
&= \mu(4) \frac{1}{x} - \frac{\left(\frac{\pi}{2}\right)^3}{4 \cdot 2!} E_1 + \frac{B_1}{2 \cdot 1!} \mu(2) x - \frac{B_2 E_0}{2 \cdot 4 \cdot 3!} x^3 - \frac{B_3 E_1}{2 \cdot 6 \cdot 5!} x^5 - \dots \quad (a = -3) \\
&= \mu(6) \frac{1}{x} - \frac{\left(\frac{\pi}{2}\right)^5}{4 \cdot 4!} E_2 + \frac{B_1}{2 \cdot 1!} \mu(4) x - \frac{B_2}{4 \cdot 3!} \mu(2) x^3 + \frac{B_3 E_0}{2 \cdot 6 \cdot 5!} x^5 + \frac{B_4 E_1}{2 \cdot 8 \cdot 7!} x^7 + \dots \quad (a = -5) \\
&\vdots \\
&= \frac{\left(\frac{\pi}{2}\right)^3}{2 \cdot 2!} E_1 \frac{1}{x} - \frac{1}{2} \mu(2) + \frac{\left(\frac{\pi}{2}\right) B_1 E_0}{2 \cdot 2 \cdot 0! \cdot 1!} x \quad (a = -2) \\
&= \frac{\left(\frac{\pi}{2}\right)^5}{2 \cdot 4!} E_2 \frac{1}{x} - \frac{1}{2} \mu(4) + \frac{\left(\frac{\pi}{2}\right)^3 B_1 E_1}{2 \cdot 2 \cdot 2! \cdot 1!} x - \frac{\left(\frac{\pi}{2}\right) B_2 E_0}{2 \cdot 4 \cdot 0! \cdot 3!} x^3 \quad (a = -4) \\
&= \frac{\left(\frac{\pi}{2}\right)^7}{2 \cdot 6!} E_3 \frac{1}{x} - \frac{1}{2} \mu(6) + \frac{\left(\frac{\pi}{2}\right)^5 B_1 E_2}{2 \cdot 2 \cdot 4! \cdot 1!} x - \frac{\left(\frac{\pi}{2}\right)^3 B_2 E_1}{2 \cdot 4 \cdot 2! \cdot 3!} x^3 + \frac{\left(\frac{\pi}{2}\right) B_3 E_0}{2 \cdot 6 \cdot 0! \cdot 5!} x^5 \quad (a = -6) \\
&\vdots
\end{aligned}$$

(78)

$$\begin{aligned}
& \frac{1}{e^x + 1} + \frac{2^a}{e^{2x} + 1} + \frac{3^a}{e^{3x} + 1} + \frac{4^a}{e^{4x} + 1} + \frac{5^a}{e^{5x} + 1} + \cdots \\
&= \frac{\pi^2}{2} B_1 \frac{1}{x^2} - \frac{B_1}{2 \cdot 2} \quad (a = 1) \\
&= \frac{(2^3 - 1)\pi^4}{4} B_2 \frac{1}{x^4} + \frac{B_2}{2 \cdot 4} \quad (a = 3) \\
&= \frac{(2^5 - 1)\pi^6}{6} B_3 \frac{1}{x^6} - \frac{B_3}{2 \cdot 6} \quad (a = 5) \\
&\vdots \\
&= \ln 2 \frac{1}{x} - \frac{1}{4} + \frac{(2^2 - 1)B_1 B_1}{2 \cdot 2 \cdot 1!} x + \frac{(2^4 - 1)B_2 B_2}{4 \cdot 4 \cdot 3!} x^3 + \frac{(2^6 - 1)B_3 B_3}{6 \cdot 6 \cdot 5!} x^5 + \cdots \quad (a = 0) \\
&= \frac{(2^2 - 1)2!}{2^2} \zeta(3) \frac{1}{x^3} - \frac{(2^2 - 1)B_1 B_2}{2 \cdot 4 \cdot 1!} x - \frac{(2^4 - 1)B_2 B_3}{4 \cdot 6 \cdot 3!} x^3 - \frac{(2^6 - 1)B_3 B_4}{6 \cdot 8 \cdot 5!} x^5 - \cdots \quad (= 2) \\
&= \frac{(2^4 - 1)4!}{2^4} \zeta(5) \frac{1}{x^5} + \frac{(2^2 - 1)B_1 B_3}{2 \cdot 6 \cdot 1!} x + \frac{(2^4 - 1)B_2 B_4}{4 \cdot 8 \cdot 3!} x^3 + \frac{(2^6 - 1)B_3 B_5}{6 \cdot 10 \cdot 5!} x^5 + \cdots \quad (a = 4) \\
&\vdots \\
&= \frac{1}{2} \zeta(3) - \frac{(2^2 - 1)(2\pi)^2}{2 \cdot 2 \cdot 1! \cdot 2!} B_1 B_1 x + \frac{2^3 - 1}{2^3 \pi^2} \zeta(3) x^2 - \frac{(2^4 - 1)B_2}{2 \cdot 4 \cdot 3!} x^3 \quad (a = -3) \\
&= \frac{1}{2} \zeta(5) - \frac{(2^2 - 1)(2\pi)^4}{2 \cdot 2 \cdot 1! \cdot 4!} B_1 B_2 x + \frac{(2^4 - 1)(2\pi)^2}{2 \cdot 4 \cdot 3! \cdot 2!} B_2 B_1 x^3 - \frac{2^5 - 1}{2^5 \pi^4} \zeta(5) x^4 + \frac{(2^6 - 1)B_3}{2 \cdot 6 \cdot 5!} x^5 \quad (a = -5) \\
&= \frac{1}{2} \zeta(7) - \frac{(2^2 - 1)(2\pi)^6}{2 \cdot 2 \cdot 1! \cdot 6!} B_1 B_3 x + \frac{(2^4 - 1)(2\pi)^4}{2 \cdot 4 \cdot 3! \cdot 4!} B_2 B_2 x^3 - \frac{(2^6 - 1)(2\pi)^2 B_3 B_1}{2 \cdot 6 \cdot 5! \cdot 2!} x^5 + \frac{2^7 - 1}{2^7 \pi^6} \zeta(7) x^6 - \frac{(2^8 - 1)B_4}{2 \cdot 8 \cdot 7!} x^7 \quad (a = -7) \\
&\vdots
\end{aligned} \tag{79}$$

$$\begin{aligned}
& \frac{1}{e^x + 1} - \frac{2^a}{e^{2x} + 1} + \frac{3^a}{e^{3x} + 1} - \frac{4^a}{e^{4x} + 1} + \frac{5^a}{e^{5x} + 1} - \dots \\
&= \frac{2^2 - 1}{2 \cdot 2} B_1 \quad (a = 1) \\
&= -\frac{2^4 - 1}{2 \cdot 4} B_2 \quad (a = 3) \\
&= \frac{2^6 - 1}{2 \cdot 6} B_3 \quad (a = 5) \\
&\vdots \\
&= \frac{1}{4} - \frac{(2^2 - 1)^2}{2 \cdot 2 \cdot 1!} B_1 B_1 x - \frac{(2^4 - 1)^2}{4 \cdot 4 \cdot 3!} B_2 B_2 x^3 - \frac{(2^6 - 1)^2}{6 \cdot 6 \cdot 5!} B_3 B_3 x^5 - \dots \quad (a = 0) \\
&= \frac{(2^4 - 1)(2^2 - 1)}{2 \cdot 4 \cdot 1!} B_1 B_2 x + \frac{(2^6 - 1)(2^4 - 1)}{4 \cdot 6 \cdot 3!} B_2 B_3 x^3 + \frac{(2^8 - 1)(2^6 - 1)}{6 \cdot 8 \cdot 5!} B_3 B_4 x^5 + \dots \quad (a = 2) \\
&= -\frac{(2^6 - 1)(2^2 - 1)}{2 \cdot 6 \cdot 1!} B_1 B_3 x - \frac{(2^8 - 1)(2^4 - 1)}{4 \cdot 8 \cdot 3!} B_2 B_4 x^3 - \frac{(2^{10} - 1)(2^6 - 1)}{6 \cdot 10 \cdot 5!} B_3 B_5 x^5 - \dots \quad (a = 4) \\
&\vdots \\
&= \frac{\ln 2}{2} - \frac{2^2 - 1}{2 \cdot 2 \cdot 1!} B_1 x \quad (a = -1) \\
&= \frac{2^2 - 1}{2^3} \zeta(3) - \frac{(2^2 - 1)\pi^2}{2 \cdot 1! \cdot 2!} B_1 B_1 x + \frac{(2^4 - 1)B_2}{2 \cdot 4 \cdot 3!} x^3 \quad (a = -3) \\
&= \frac{2^4 - 1}{2^5} \zeta(5) - \frac{(2^3 - 1)(2^2 - 1)\pi^4}{2 \cdot 1! \cdot 4!} B_2 B_1 x + \frac{(2 - 1)(2^4 - 1)\pi^2}{4 \cdot 3! \cdot 2!} B_1 B_2 x^3 - \frac{(2^6 - 1)B_3}{2 \cdot 6 \cdot 5!} x^5 \quad (a = -5) \\
&= \frac{2^6 - 1}{2^7} \zeta(7) - \frac{(2^5 - 1)(2^2 - 1)\pi^6}{2 \cdot 1! \cdot 6!} B_1 B_3 x + \frac{(2^3 - 1)(2^4 - 1)\pi^4}{4 \cdot 3! \cdot 4!} B_2 B_2 x^3 - \frac{(2 - 1)(2^6 - 1)\pi^2}{6 \cdot 5! \cdot 2!} B_3 B_1 x^5 \\
&\quad + \frac{(2^8 - 1)B_4}{2 \cdot 8 \cdot 7!} x^7 \quad (a = -7) \\
&\vdots \\
&= \frac{\pi^2}{2 \cdot 2!} B_1 - \frac{(2^2 - 1)\ln 2}{2 \cdot 1!} B_1 x + \frac{(2^2 - 1)(2^4 - 1)}{2 \cdot 4 \cdot 3!} B_2 B_1 x^3 + \frac{(2^4 - 1)(2^6 - 1)}{4 \cdot 6 \cdot 5!} B_3 B_2 x^5 + \dots \quad (a = -2) \\
&= \frac{(2^3 - 1)\pi^4}{2 \cdot 4!} B_2 - \frac{(2^2 - 1)(2^2 - 1)B_1}{2 \cdot 2^2 \cdot 1!} \zeta(3)x + \frac{(2^4 - 1)\ln 2}{4 \cdot 3!} B_2 x^3 - \frac{(2^2 - 1)(2^6 - 1)}{2 \cdot 6 \cdot 5!} B_1 B_3 x^5 \\
&\quad - \frac{(2^4 - 1)(2^8 - 1)}{4 \cdot 8 \cdot 7!} B_2 B_4 x^7 - \dots \quad (a = -4) \\
&= \frac{(2^5 - 1)\pi^6}{2 \cdot 6!} B_3 - \frac{(2^4 - 1)(2^2 - 1)B_1}{2 \cdot 2^4 \cdot 1!} \zeta(5)x + \frac{(2^2 - 1)(2^4 - 1)B_2}{4 \cdot 2^2 \cdot 3!} \zeta(3)x^3 - \frac{(2^6 - 1)\ln 2}{6 \cdot 5!} B_3 x^5 + \frac{(2^2 - 1)(2^8 - 1)}{2 \cdot 8 \cdot 7!} B_1 B_4 x^7 \\
&\quad + \frac{(2^4 - 1)(2^{10} - 1)}{4 \cdot 10 \cdot 9!} B_2 B_5 x^9 + \dots \quad (a = -6) \\
&\vdots
\end{aligned} \tag{80}$$

$$\begin{aligned}
& \frac{1}{e^x + 1} + \frac{3^a}{e^{3x} + 1} + \frac{5^a}{e^{5x} + 1} + \frac{7^a}{e^{7x} + 1} + \cdots \\
&= \frac{\pi^2}{2 \cdot 2} B_1 \frac{1}{x^2} + \frac{B_1}{2 \cdot 2} \quad (a = 1) \\
&= \frac{(2^3 - 1)\pi^4}{2 \cdot 4} B_2 \frac{1}{x^4} - \frac{2^3 - 1}{2 \cdot 4} B_2 \quad (a = 3) \\
&= \frac{(2^5 - 1)\pi^6}{2 \cdot 6} B_3 \frac{1}{x^6} + \frac{2^5 - 1}{2 \cdot 6} B_3 \quad (a = 5) \\
&\vdots \\
&= \frac{\ln 2}{2} \frac{1}{x} - \frac{(2 - 1)(2^2 - 1)}{2 \cdot 2 \cdot 1!} B_1 B_1 x - \frac{(2^3 - 1)(2^4 - 1)}{4 \cdot 4 \cdot 3!} B_2 B_2 x^3 - \cdots \quad (a = 0) \\
&= \frac{(2^2 - 1)2!}{2^3} \zeta(3) \frac{1}{x^3} + \frac{(2^3 - 1)(2^2 - 1)}{2 \cdot 4 \cdot 1!} B_2 B_1 x + \frac{(2^5 - 1)(2^4 - 1)}{4 \cdot 6 \cdot 3!} B_3 B_2 x^3 + \cdots \quad (a = 2) \\
&= \frac{(2^4 - 1)4!}{2^5} \zeta(5) \frac{1}{x^5} - \frac{(2^5 - 1)(2^2 - 1)}{2 \cdot 6 \cdot 1!} B_3 B_1 x - \frac{(2^7 - 1)(2^4 - 1)}{4 \cdot 8 \cdot 3!} B_4 B_2 x^3 - \cdots \quad (a = 4) \\
&= \frac{(2^6 - 1)6!}{2^7} \zeta(7) \frac{1}{x^7} + \frac{(2^7 - 1)(2^2 - 1)}{2 \cdot 8 \cdot 1!} B_4 B_1 x + \frac{(2^9 - 1)(2^4 - 1)}{4 \cdot 10 \cdot 3!} B_5 B_2 x^3 + \cdots \quad (a = 6) \\
&\vdots \\
&= \frac{2^3 - 1}{2^4} \zeta(3) - \frac{(2^2 - 1)^2 \pi^2}{2 \cdot 2 \cdot 1! \cdot 2!} B_1 B_1 x + \frac{2^3 - 1}{2^4 \pi^2} \zeta(3) x^2 \quad (a = -3) \\
&= \frac{2^5 - 1}{2^6} \zeta(5) - \frac{(2^4 - 1)(2^2 - 1) \pi^4}{2 \cdot 2 \cdot 1! \cdot 4!} B_2 B_1 x + \frac{(2^2 - 1)(2^4 - 1) \pi^2}{2 \cdot 4 \cdot 3! \cdot 2!} B_1 B_2 x^3 - \frac{2^5 - 1}{2^6 \pi^4} \zeta(5) x^4 \quad (a = -5) \\
&= \frac{2^7 - 1}{2^8} \zeta(7) - \frac{(2^6 - 1)(2^2 - 1) \pi^6}{2 \cdot 2 \cdot 1! \cdot 6!} B_3 B_1 x + \frac{(2^4 - 1)(2^4 - 1) \pi^4}{2 \cdot 4 \cdot 3! \cdot 4!} B_2 B_2 x^3 - \frac{(2^2 - 1)(2^6 - 1) \pi^2}{2 \cdot 6 \cdot 5! \cdot 2!} B_1 B_3 x^5 \\
&\quad + \frac{2^7 - 1}{2^8 \pi^6} \zeta(7) x^6 \quad (a = -7) \\
&\vdots
\end{aligned} \tag{81}$$

$$\begin{aligned}
& \frac{1}{e^x + 1} - \frac{3^a}{e^{3x} + 1} + \frac{5^a}{e^{5x} + 1} - \frac{7^a}{e^{7x} + 1} + \cdots \\
&= \frac{2^2 - 1}{2 \cdot 2 \cdot 1!} E_1 B_1 x + \frac{2^4 - 1}{2 \cdot 4 \cdot 3!} E_2 B_2 x^3 + \frac{2^6 - 1}{2 \cdot 6 \cdot 5!} E_3 B_3 x^5 + \cdots \quad (a = 1) \\
&= -\frac{2^2 - 1}{2 \cdot 2 \cdot 1!} B_1 E_2 x - \frac{2^4 - 1}{2 \cdot 4 \cdot 3!} B_2 E_3 x^3 - \frac{2^6 - 1}{2 \cdot 6 \cdot 5!} B_3 E_4 x^5 - \cdots \quad (a = 3) \\
&= \frac{2^2 - 1}{2 \cdot 2 \cdot 1!} E_3 B_1 x + \frac{2^4 - 1}{2 \cdot 4 \cdot 3!} E_4 B_2 x^3 + \frac{2^6 - 1}{2 \cdot 6 \cdot 5!} E_5 B_3 x^5 + \cdots \quad (a = 5) \\
&\vdots \\
&= \frac{E_0}{2 \cdot 2} \quad (a = 0) \\
&= -\frac{E_1}{2 \cdot 2} \quad (a = 2) \\
&= \frac{E_2}{2 \cdot 2} \quad (a = 4) \\
&= -\frac{E_3}{2 \cdot 2} \quad (a = 6) \\
&\vdots \\
&= \frac{\pi}{2^3 \cdot 0!} E_0 - \frac{2^2 - 1}{2 \cdot 2 \cdot 1!} B_1 E_0 x - \frac{2^4 - 1}{2 \cdot 4 \cdot 3!} B_2 E_1 x^3 - \frac{2^6 - 1}{2 \cdot 6 \cdot 5!} B_3 E_2 x^5 - \cdots \quad (a = -1) \\
&= \frac{\pi^3}{2^5 \cdot 2!} E_1 - \frac{2^2 - 1}{2 \cdot 1!} B_1 \mu(2) x + \frac{2^4 - 1}{2 \cdot 4 \cdot 3!} B_2 E_0 x^3 + \frac{2^6 - 1}{2 \cdot 6 \cdot 5!} B_3 E_1 x^5 + \cdots \quad (a = -3) \\
&= \frac{\pi^5}{2^7 \cdot 4!} E_2 - \frac{2^2 - 1}{2 \cdot 1!} B_1 \mu(4) x + \frac{2^4 - 1}{4 \cdot 3!} B_2 \mu(2) x^3 - \frac{2^6 - 1}{2 \cdot 6 \cdot 5!} B_3 E_0 x^5 - \frac{2^8 - 1}{2 \cdot 8 \cdot 7!} B_4 E_1 x^7 - \cdots \quad (a = -5) \\
&\vdots \\
&= \frac{1}{2} \mu(2) - \frac{(2^2 - 1) \left(\frac{\pi}{2}\right)}{2 \cdot 2 \cdot 1! \cdot 0!} B_1 E_0 x \quad (a = -2) \\
&= \frac{1}{2} \mu(4) - \frac{(2^2 - 1) \left(\frac{\pi}{2}\right)^3}{2 \cdot 2 \cdot 1! \cdot 2!} B_1 E_1 x + \frac{(2^4 - 1) \left(\frac{\pi}{2}\right)}{2 \cdot 4 \cdot 3! \cdot 0!} B_2 E_0 x^3 \quad (a = -4) \\
&= \frac{1}{2} \mu(6) - \frac{(2^2 - 1) \left(\frac{\pi}{2}\right)^5}{2 \cdot 2 \cdot 1! \cdot 4!} B_1 E_2 x + \frac{(2^4 - 1) \left(\frac{\pi}{2}\right)^3}{2 \cdot 4 \cdot 3! \cdot 2!} B_2 E_1 x^3 - \frac{(2^6 - 1) \left(\frac{\pi}{2}\right)}{2 \cdot 6 \cdot 5! \cdot 0!} B_3 E_0 x^5 \quad (a = -6) \\
&\vdots
\end{aligned} \tag{82}$$

$$\begin{aligned}
& \frac{1}{e^x - e^{-x}} + \frac{2^a}{e^{2x} - e^{-2x}} + \frac{3^a}{e^{3x} - e^{-3x}} + \frac{4^a}{e^{4x} - e^{-4x}} + \frac{5^a}{e^{5x} - e^{-5x}} + \dots \\
&= \frac{(2^2 - 1)\pi^2}{2 \cdot 2} B_1 \frac{1}{x^2} - \frac{1}{4x} \quad (a = 1) \\
&= \frac{(2^4 - 1)\pi^4}{2 \cdot 4} B_2 \frac{1}{x^4} \quad (a = 3) \\
&= \frac{(2^6 - 1)\pi^6}{2 \cdot 6} B_3 \frac{1}{x^6} \quad (a = 5) \\
&= \frac{(2^8 - 1)\pi^8}{2 \cdot 8} B_4 \frac{1}{x^8} \quad (a = 7) \\
&\vdots \\
&= \frac{(2^3 - 1)2!}{2^3} \zeta(3) \frac{1}{x^3} - \frac{B_1}{2 \cdot 2} \frac{1}{x} - \frac{B_1 B_2}{2 \cdot 4 \cdot 1!} x - \frac{(2^3 - 1)B_2 B_3}{4 \cdot 6 \cdot 3!} x^3 - \frac{(2^5 - 1)B_3 B_4}{6 \cdot 8 \cdot 5!} x^5 - \dots \quad (a = 2) \\
&= \frac{(2^5 - 1)4!}{2^5} \zeta(5) \frac{1}{x^5} + \frac{B_2}{2 \cdot 4} \frac{1}{x} + \frac{B_1 B_3}{2 \cdot 6 \cdot 1!} x + \frac{(2^3 - 1)B_2 B_4}{4 \cdot 8 \cdot 3!} x^3 + \frac{(2^5 - 1)B_3 B_5}{6 \cdot 10 \cdot 5!} x^5 + \dots \quad (a = 4) \\
&= \frac{(2^7 - 1)6!}{2^7} \zeta(7) \frac{1}{x^7} - \frac{B_3}{2 \cdot 6} \frac{1}{x} - \frac{B_1 B_4}{2 \cdot 8 \cdot 1!} x - \frac{(2^3 - 1)B_2 B_5}{4 \cdot 10 \cdot 3!} x^3 - \frac{(2^5 - 1)B_3 B_6}{6 \cdot 12 \cdot 5!} x^5 - \dots \quad (a = 6) \\
&\vdots \\
&= \frac{\pi^2}{2!} B_1 \frac{1}{x} - \frac{\ln 2}{2} + \frac{B_1}{2 \cdot 2 \cdot 1!} x \quad (a = -1) \\
&= \frac{2^2 \pi^4}{4!} B_2 \frac{1}{x} - \frac{(2-1)(2\pi)^2}{2 \cdot 2 \cdot 1! \cdot 2!} B_1 B_1 x + \frac{2^2 - 1}{2(2\pi)^2} \zeta(3) x^2 - \frac{2^3 - 1}{2 \cdot 4 \cdot 3!} B_2 x^3 \quad (a = -3) \\
&= \frac{2^4 \pi^6}{6!} B_3 \frac{1}{x} - \frac{(2-1)(2\pi)^4}{2 \cdot 2 \cdot 1! \cdot 4!} B_1 B_2 x + \frac{(2^3 - 1)(2\pi)^2}{2 \cdot 4 \cdot 3! \cdot 2!} B_2 B_1 x^3 - \frac{2^4 - 1}{2(2\pi)^4} \zeta(5) x^4 + \frac{2^5 - 1}{2 \cdot 6 \cdot 5!} B_3 x^5 \quad (a = -5) \\
&= \frac{2^6 \pi^8}{8!} B_4 \frac{1}{x} - \frac{(2-1)(2\pi)^6}{2 \cdot 2 \cdot 1! \cdot 6!} B_1 B_3 x + \frac{(2^3 - 1)(2\pi)^4}{2 \cdot 4 \cdot 3! \cdot 4!} B_2 B_2 x^3 - \frac{(2^5 - 1)(2\pi)^2}{2 \cdot 6 \cdot 5 \cdot 2!} B_3 B_1 x^5 + \frac{2^6 - 1}{2(2\pi)^6} \zeta(7) x^6 \\
&\quad - \frac{2^7 - 1}{2 \cdot 8 \cdot 7!} B_4 x^7 \quad (a = -7) \\
&\vdots
\end{aligned} \tag{83}$$

$$\begin{aligned}
& \frac{1}{e^x - e^{-x}} - \frac{2^a}{e^{2x} - e^{-2x}} + \frac{3^a}{e^{3x} - e^{-3x}} - \frac{4^a}{e^{4x} - e^{-4x}} + \dots \\
&= \frac{\ln 2}{2} \frac{1}{x} - \frac{(2-1)(2^2-1)}{2 \cdot 2 \cdot 1!} B_1 B_1 x - \frac{(2^3-1)(2^4-1)}{4 \cdot 4 \cdot 3!} B_2 B_2 x^3 - \frac{(2^5-1)(2^6-1)}{6 \cdot 6 \cdot 5!} B_3 B_3 x^5 - \dots \quad (a=0) \\
&= \frac{2^2-1}{2 \cdot 2} B_1 \frac{1}{x} + \frac{(2-1)(2^4-1)}{2 \cdot 4 \cdot 1!} B_1 B_2 x + \frac{(2^3-1)(2^6-1)}{4 \cdot 6 \cdot 3!} B_2 B_3 x^3 + \frac{(2^5-1)(2^8-1)}{6 \cdot 8 \cdot 5!} B_3 B_4 x^5 + \dots \quad (a=2) \\
&= -\frac{2^4-1}{2 \cdot 4} B_2 \frac{1}{x} - \frac{(2-1)(2^6-1)}{2 \cdot 6 \cdot 1!} B_1 B_3 x - \frac{(2^3-1)(2^8-1)}{4 \cdot 8 \cdot 3!} B_2 B_4 x^3 - \frac{(2^5-1)(2^{10}-1)}{6 \cdot 10 \cdot 5!} B_3 B_5 x^5 - \dots \quad (a=4) \\
&\vdots \\
&= \frac{1}{4x} \quad (a=1) \\
&= 0 \quad (a=3, 5, 7, 9, \dots) \\
&= \frac{\pi^2}{2 \cdot 2!} B_1 \frac{1}{x} - \frac{B_1}{2 \cdot 2 \cdot 1!} x \quad (a=-1) \\
&= \frac{(2^3-1)\pi^4}{2 \cdot 4!} B_2 \frac{1}{x} - \frac{(2-1)(2-1)\pi^2}{2 \cdot 1! \cdot 2!} B_1 B_1 x + \frac{2^3-1}{2 \cdot 4 \cdot 3!} B_2 x^3 \quad (a=-3) \\
&= \frac{(2^5-1)\pi^6}{2 \cdot 6!} B_3 \frac{1}{x} - \frac{(2-1)(2^3-1)\pi^4}{2 \cdot 1! \cdot 4!} B_1 B_2 x + \frac{(2^3-1)(2-1)\pi^2}{4 \cdot 3! \cdot 2!} B_2 B_1 x^3 - \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 x^5 \quad (a=-5) \\
&= \frac{(2^7-1)\pi^8}{2 \cdot 8!} B_4 \frac{1}{x} - \frac{(2-1)(2^5-1)\pi^6}{2 \cdot 1! \cdot 6!} B_1 B_3 x + \frac{(2^3-1)(2^3-1)\pi^4}{4 \cdot 3! \cdot 4!} B_2 B_2 x^3 - \frac{(2^5-1)(2-1)\pi^2}{6 \cdot 5! \cdot 2!} B_3 B_1 x^5 \\
&\quad + \frac{2^7-1}{2 \cdot 8 \cdot 7!} B_4 x^7 \quad (a=-7) \\
&\vdots \\
&= \frac{2^2-1}{2^3} \zeta(3) \frac{1}{x} - \frac{(2-1)\ln 2}{2 \cdot 1!} B_1 x + \frac{(2^3-1)(2^2-1)}{2 \cdot 4 \cdot 3!} B_2 B_1 x^3 + \frac{(2^5-1)(2^4-1)}{4 \cdot 6 \cdot 5!} B_3 B_2 x^5 \\
&\quad + \frac{(2^7-1)(2^6-1)}{6 \cdot 8 \cdot 7!} B_4 B_3 x^7 + \dots \quad (a=-2) \\
&= \frac{2^4-1}{2^5} \zeta(5) \frac{1}{x} - \frac{(2-1)(2^2-1)}{2^3 \cdot 1!} B_1 \zeta(3) x + \frac{(2^3-1)\ln 2}{4 \cdot 3!} B_2 x^3 - \frac{(2^5-1)(2^2-1)}{2 \cdot 6 \cdot 5!} B_3 B_1 x^5 \\
&\quad - \frac{(2^7-1)(2^4-1)}{4 \cdot 8 \cdot 7!} B_4 B_2 x^7 - \dots \quad (a=-4) \\
&= \frac{2^6-1}{2^7} \zeta(7) \frac{1}{x} - \frac{(2-1)(2^4-1)}{2 \cdot 2^4 \cdot 1!} B_1 \zeta(5) x + \frac{(2^3-1)(2^2-1)}{4 \cdot 2^2 \cdot 3!} B_2 \zeta(3) x^3 - \frac{(2^5-1)\ln 2}{6 \cdot 5!} B_3 x^5 \\
&\quad + \frac{(2^7-1)(2^2-1)}{2 \cdot 8 \cdot 7!} B_4 B_1 x^7 + \frac{(2^9-1)(2^4-1)}{4 \cdot 10 \cdot 9!} B_5 B_2 x^9 + \dots \quad (a=-6) \\
&= \frac{2^8-1}{2^9} \zeta(9) \frac{1}{x} - \frac{(2-1)(2^6-1)}{2 \cdot 2^6 \cdot 1!} B_1 \zeta(7) x + \frac{(2^3-1)(2^4-1)}{4 \cdot 2^4 \cdot 3!} B_2 \zeta(5) x^3 - \frac{(2^5-1)(2^2-1)}{6 \cdot 2^2 \cdot 5!} B_3 \zeta(3) x^5 \\
&\quad + \frac{(2^7-1)\ln 2}{8 \cdot 7!} B_4 x^7 - \frac{(2^9-1)(2^2-1)}{2 \cdot 10 \cdot 9!} B_5 B_1 x^9 - \dots \quad (a=-8) \\
&\vdots
\end{aligned} \tag{84}$$

$$\begin{aligned}
& \frac{1}{e^x - e^{-x}} + \frac{3^a}{e^{3x} - e^{-3x}} + \frac{5^a}{e^{5x} - e^{-5x}} + \frac{7^a}{e^{7x} - e^{-7x}} + \cdots \\
&= \frac{(2^2 - 1)\pi^2}{4 \cdot 2} B_1 \frac{1}{x^2} \quad (a = 1) \\
&= \frac{(2^4 - 1)\pi^4}{4 \cdot 4} B_2 \frac{1}{x^4} \quad (a = 3) \\
&= \frac{(2^6 - 1)\pi^6}{4 \cdot 6} B_3 \frac{1}{x^6} \quad (a = 5) \\
&\vdots \\
&= \frac{(2^3 - 1)2!}{2^4} \zeta(3) \frac{1}{x^3} + \frac{B_1}{2 \cdot 2} \frac{1}{x} + \frac{(2 - 1)(2^3 - 1)}{2 \cdot 4 \cdot 1!} B_1 B_2 x + \frac{(2^3 - 1)(2^5 - 1)}{4 \cdot 6 \cdot 3!} B_2 B_3 x^3 \\
&\quad + \frac{(2^5 - 1)(2^7 - 1)}{6 \cdot 8 \cdot 5!} B_3 B_4 x^5 + \cdots \quad (a = 2) \\
&= \frac{(2^5 - 1)4!}{2^6} \zeta(5) \frac{1}{x^5} - \frac{2^3 - 1}{2 \cdot 4} B_2 \frac{1}{x} - \frac{(2 - 1)(2^5 - 1)}{2 \cdot 6 \cdot 1!} B_1 B_3 x - \frac{(2^3 - 1)(2^7 - 1)}{4 \cdot 8 \cdot 3!} B_2 B_4 x^3 \\
&\quad - \frac{(2^5 - 1)(2^9 - 1)}{6 \cdot 10 \cdot 5!} B_3 B_5 x^5 - \cdots \quad (a = 4) \\
&= \frac{(2^7 - 1)6!}{2^8} \zeta(7) \frac{1}{x^7} + \frac{2^5 - 1}{2 \cdot 6} B_3 \frac{1}{x} + \frac{(2 - 1)(2^7 - 1)}{2 \cdot 8 \cdot 1!} B_1 B_4 x + \frac{(2^3 - 1)(2^9 - 1)}{4 \cdot 10 \cdot 3!} B_2 B_5 x^3 \\
&\quad + \frac{(2^5 - 1)(2^{11} - 1)}{6 \cdot 12 \cdot 5!} B_3 B_6 x^5 + \cdots \quad (a = 6) \\
&\vdots \\
&= \frac{(2^2 - 1)\pi^2}{4 \cdot 2!} B_1 \frac{1}{x} - \frac{\ln 2}{4} \quad (a = -1) \\
&= \frac{(2^4 - 1)\pi^4}{4 \cdot 4!} B_2 \frac{1}{x} - \frac{(2 - 1)(2^2 - 1)\pi^2}{2 \cdot 2 \cdot 1! \cdot 2!} B_1 B_1 x + \frac{2^2 - 1}{2^4 \pi^2} \zeta(3) x^2 \quad (a = -1) \\
&= \frac{(2^6 - 1)\pi^6}{4 \cdot 6!} B_3 \frac{1}{x} - \frac{(2 - 1)(2^4 - 1)\pi^4}{2 \cdot 2 \cdot 1! \cdot 4!} B_1 B_2 x + \frac{(2^3 - 1)(2^2 - 1)\pi^2}{2 \cdot 4 \cdot 3! \cdot 2!} B_2 B_1 x^3 - \frac{2^4 - 1}{2^6 \pi^4} \zeta(5) x^4 \quad (a = -3) \\
&= \frac{(2^8 - 1)\pi^8}{4 \cdot 8!} B_4 \frac{1}{x} - \frac{(2 - 1)(2^6 - 1)\pi^6}{2 \cdot 2 \cdot 1! \cdot 6!} B_1 B_3 x + \frac{(2^3 - 1)(2^4 - 1)\pi^4}{2 \cdot 4 \cdot 3! \cdot 4!} B_2 B_2 x^3 - \frac{(2^5 - 1)(2^2 - 1)\pi^2}{2 \cdot 6 \cdot 5! \cdot 2!} B_3 B_1 x^5 \\
&\quad + \frac{2^6 - 1}{2^8 \pi^6} \zeta(7) x^6 \quad (a = -7) \\
&\vdots
\end{aligned} \tag{85}$$

$$\begin{aligned}
& \frac{1}{e^x - e^{-x}} - \frac{3^a}{e^{3x} - e^{-3x}} + \frac{5^a}{e^{5x} - e^{-5x}} - \frac{7^a}{e^{7x} - e^{-7x}} + \cdots \\
&= \frac{E_0}{4} \frac{1}{x} + \frac{2-1}{2 \cdot 2 \cdot 1!} B_1 E_1 x + \frac{2^3-1}{2 \cdot 4 \cdot 3!} B_2 E_2 x^3 + \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 E_3 x^5 + \cdots \quad (a=1) \\
&= -\frac{E_1}{4} \frac{1}{x} - \frac{2-1}{2 \cdot 2 \cdot 1!} B_1 E_2 x - \frac{2^3-1}{2 \cdot 4 \cdot 3!} B_2 E_3 x^3 - \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 E_4 x^5 - \cdots \quad (a=3) \\
&= \frac{E_2}{4} \frac{1}{x} + \frac{2-1}{2 \cdot 2 \cdot 1!} B_1 E_3 x + \frac{2^3-1}{2 \cdot 4 \cdot 3!} B_2 E_4 x^3 + \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 E_5 x^5 + \cdots \quad (a=5) \\
&\vdots \\
&= \frac{\pi}{8} \frac{1}{x} \quad (a=0) \\
&= 0 \quad (a=2, 4, 6, 8, \dots) \\
&= \frac{1}{2} \mu(2) \frac{1}{x} - \frac{2-1}{2 \cdot 2 \cdot 1!} B_1 E_0 x - \frac{2^3-1}{2 \cdot 4 \cdot 3!} B_2 E_1 x^3 - \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 E_2 x^5 - \cdots \quad (a=-1) \\
&= \frac{1}{2} \mu(4) \frac{1}{x} - \frac{2-1}{2 \cdot 1!} B_1 \mu(2) x + \frac{2^3-1}{2 \cdot 4 \cdot 3!} B_2 E_0 x^3 + \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 E_1 x^5 + \cdots \quad (a=-3) \\
&= \frac{1}{2} \mu(6) \frac{1}{x} - \frac{2-1}{2 \cdot 1!} B_1 \mu(4) x + \frac{2^3-1}{4 \cdot 3!} B_2 \mu(2) x^3 - \frac{2^5-1}{2 \cdot 6 \cdot 5!} B_3 E_0 x^5 - \cdots \quad (a=-5) \\
&\vdots \\
&= \frac{\left(\frac{\pi}{2}\right)^3}{4 \cdot 2!} E_1 \frac{1}{x} - \frac{(2-1) \left(\frac{\pi}{2}\right)}{2 \cdot 2 \cdot 0! \cdot 1!} B_1 E_0 x \quad (a=-2) \\
&= \frac{\left(\frac{\pi}{2}\right)^5}{4 \cdot 4!} E_2 \frac{1}{x} - \frac{(2-1) \left(\frac{\pi}{2}\right)^3}{2 \cdot 2 \cdot 1! \cdot 2!} B_1 E_1 x + \frac{(2^3-1) \left(\frac{\pi}{2}\right)}{2 \cdot 4 \cdot 3! \cdot 0!} B_2 E_0 x^3 \quad (a=-4) \\
&= \frac{\left(\frac{\pi}{2}\right)^7}{4 \cdot 6!} E_3 \frac{1}{x} - \frac{(2-1) \left(\frac{\pi}{2}\right)^5}{2 \cdot 2 \cdot 1! \cdot 4!} B_1 E_2 x + \frac{(2^3-1) \left(\frac{\pi}{2}\right)^3}{2 \cdot 4 \cdot 3! \cdot 2!} B_2 E_1 x^3 - \frac{(2^5-1) \left(\frac{\pi}{2}\right)}{2 \cdot 6 \cdot 5! \cdot 0!} B_3 E_0 x^5 \quad (a=-6) \\
&= \frac{\left(\frac{\pi}{2}\right)^9}{4 \cdot 8!} E_4 \frac{1}{x} - \frac{(2-1) \left(\frac{\pi}{2}\right)^7}{2 \cdot 2 \cdot 1! \cdot 6!} B_1 E_3 x + \frac{(2^3-1) \left(\frac{\pi}{2}\right)^5}{2 \cdot 4 \cdot 3! \cdot 4!} B_2 E_2 x^3 - \frac{(2^5-1) \left(\frac{\pi}{2}\right)^3}{2 \cdot 6 \cdot 5! \cdot 2!} B_3 E_1 x^5 + \frac{(2^7-1) \left(\frac{\pi}{2}\right)}{2 \cdot 8 \cdot 7! \cdot 0!} B_4 E_0 x^7 \quad (a=-8) \\
&\vdots
\end{aligned} \tag{86}$$

$$\begin{aligned}
& \frac{1}{e^x + e^{-x}} + \frac{2^a}{e^{2x} + e^{-2x}} + \frac{3^a}{e^{3x} + e^{-3x}} + \frac{4^a}{e^{4x} + e^{-4x}} + \frac{5^a}{e^{5x} + e^{-5x}} + \dots \\
&= 1! \cdot \mu(2) \frac{1}{x^2} - \frac{E_0 B_1}{2 \cdot 2 \cdot 0!} - \frac{E_1 B_2}{2 \cdot 4 \cdot 2!} x^2 - \frac{E_2 B_3}{2 \cdot 6 \cdot 4!} x^4 - \dots \quad (a = 1) \\
&= 3! \cdot \mu(4) \frac{1}{x^4} + \frac{E_0 B_2}{2 \cdot 4 \cdot 0!} + \frac{E_1 B_3}{2 \cdot 6 \cdot 2!} x^2 + \frac{E_2 B_4}{2 \cdot 8 \cdot 4!} x^4 + \dots \quad (a = 3) \\
&= 5! \cdot \mu(6) \frac{1}{x^6} - \frac{E_0 B_3}{2 \cdot 6 \cdot 0!} - \frac{E_1 B_4}{2 \cdot 8 \cdot 2!} x^2 - \frac{E_2 B_5}{2 \cdot 10 \cdot 4!} x^4 - \dots \quad (a = 5) \\
&\vdots \\
&= \frac{1}{2} \left(\frac{\pi}{2} \right) E_0 \frac{1}{x} - \frac{1}{4} \quad (a = 0) \\
&= \frac{1}{2} \left(\frac{\pi}{2} \right)^3 E_1 \frac{1}{x^3} \quad (a = 2) \\
&= \frac{1}{2} \left(\frac{\pi}{2} \right)^5 E_2 \frac{1}{x^5} \quad (a = 4) \\
&= \frac{1}{2} \left(\frac{\pi}{2} \right)^7 E_3 \frac{1}{x^7} \quad (a = 6) \\
&\vdots \\
&= \frac{(2\pi)^2}{2 \cdot 2 \cdot 0! \cdot 2!} E_0 B_1 - \frac{2}{\pi} \mu(2) x + \frac{E_1}{2 \cdot 2 \cdot 2!} x^2 \quad (a = -2) \\
&= \frac{(2\pi)^4}{2 \cdot 2 \cdot 0! \cdot 4!} E_0 B_2 - \frac{(2\pi)^2}{2 \cdot 2 \cdot 2! \cdot 2!} E_1 B_1 x^2 + \left(\frac{2}{\pi} \right)^3 \mu(4) x^3 - \frac{E_2}{2 \cdot 2 \cdot 4!} x^4 \quad (a = -2) \\
&= \frac{(2\pi)^6}{2 \cdot 2 \cdot 0! \cdot 6!} E_0 B_3 - \frac{(2\pi)^4}{2 \cdot 2 \cdot 2! \cdot 4!} E_1 B_2 x^2 + \frac{(2\pi)^2}{2 \cdot 2 \cdot 4! \cdot 2!} E_2 B_1 x^4 - \left(\frac{2}{\pi} \right)^5 \mu(6) x^5 + \frac{E_3}{2 \cdot 2 \cdot 6!} x^6 \quad (a = -6) \\
&\vdots
\end{aligned} \tag{87}$$

$$\begin{aligned}
& \frac{1}{e^x + e^{-x}} - \frac{2^a}{e^{2x} + e^{-2x}} + \frac{3^a}{e^{3x} + e^{-3x}} - \frac{4^a}{e^{4x} + e^{-4x}} + \frac{5^a}{e^{5x} + e^{-5x}} - \dots \\
&= \frac{2^2 - 1}{2 \cdot 2 \cdot 0!} E_0 B_1 + \frac{2^4 - 1}{2 \cdot 4 \cdot 2!} E_1 B_2 x^2 + \frac{2^6 - 1}{2 \cdot 6 \cdot 4!} E_2 B_3 x^4 + \dots \quad (a = 1) \\
&= -\frac{2^4 - 1}{2 \cdot 4 \cdot 0!} E_0 B_2 - \frac{2^6 - 1}{2 \cdot 6 \cdot 2!} E_1 B_3 x^2 - \frac{2^8 - 1}{2 \cdot 8 \cdot 4!} E_2 B_4 x^4 - \dots \quad (a = 3) \\
&= \frac{2^6 - 1}{2 \cdot 6 \cdot 0!} E_0 B_3 + \frac{2^8 - 1}{2 \cdot 8 \cdot 2!} E_1 B_4 x^2 + \frac{2^{10} - 1}{2 \cdot 10 \cdot 4!} E_2 B_5 x^4 + \dots \quad (a = 5) \\
&\vdots \\
&= \frac{1}{4} \quad (a = 0) \\
&= 0 \quad (a = 2, 4, 6, 8, 10, \dots) \\
&= \frac{\ln 2}{2} - \frac{2^2 - 1}{2 \cdot 2 \cdot 2!} E_1 B_1 x^2 - \frac{2^4 - 1}{2 \cdot 4 \cdot 4!} E_2 B_2 x^4 - \frac{2^6 - 1}{2 \cdot 6 \cdot 6!} E_3 B_3 x^6 - \dots \quad (a = -1) \\
&= \frac{2^2 - 1}{2^3} E_0 \zeta(3) - \frac{\ln 2}{2 \cdot 2!} E_1 x^2 + \frac{2^2 - 1}{2 \cdot 2 \cdot 4!} E_2 B_1 x^4 + \frac{2^4 - 1}{2 \cdot 4 \cdot 6!} E_3 B_2 x^6 + \dots \quad (a = -3) \\
&= \frac{2^4 - 1}{2^5} E_0 \zeta(5) - \frac{2^2 - 1}{2^3 \cdot 2!} E_1 \zeta(3) x^2 + \frac{\ln 2}{2 \cdot 4!} E_2 x^4 - \frac{2^2 - 1}{2 \cdot 2 \cdot 6!} E_3 B_1 x^6 - \dots \quad (a = -5) \\
&\vdots \\
&= \frac{\pi^2}{2 \cdot 0! \cdot 2!} E_0 B_1 - \frac{E_1}{4 \cdot 2!} x^2 \quad (a = -2) \\
&= \frac{(2^3 - 1)\pi^4}{2 \cdot 0! \cdot 4!} E_0 B_2 - \frac{\pi^2}{2 \cdot 2! \cdot 2!} E_1 B_1 x^2 + \frac{E_2}{4 \cdot 4!} x^4 \quad (a = -4) \\
&= \frac{(2^5 - 1)\pi^6}{2 \cdot 0! \cdot 6!} E_0 B_3 - \frac{(2^3 - 1)\pi^4}{2 \cdot 2! \cdot 4!} E_1 B_2 x^2 + \frac{\pi^2}{2 \cdot 4! \cdot 2!} E_2 B_1 x^4 - \frac{E_3}{4 \cdot 6!} x^6 \quad (a = -6) \\
&= \frac{(2^7 - 1)\pi^8}{2 \cdot 0! \cdot 8!} E_0 B_4 - \frac{(2^5 - 1)\pi^6}{2 \cdot 2! \cdot 6!} E_1 B_3 x^2 + \frac{(2^3 - 1)\pi^4}{2 \cdot 4! \cdot 4!} E_2 B_2 x^4 - \frac{\pi^2}{2 \cdot 6! \cdot 2!} E_3 B_1 x^6 + \frac{E_4}{4 \cdot 8!} x^8 \quad (a = -8) \\
&\vdots
\end{aligned} \tag{88}$$

$$\begin{aligned}
& \frac{1}{e^x + e^{-x}} + \frac{3^a}{e^{3x} + e^{-3x}} + \frac{5^a}{e^{5x} + e^{-5x}} + \frac{7^a}{e^{7x} + e^{-7x}} + \frac{9^a}{e^{9x} + e^{-9x}} + \cdots \\
&= \frac{1}{2} \mu(2) \frac{1}{x^2} + \frac{2-1}{2 \cdot 2 \cdot 0!} E_0 B_1 + \frac{2^3-1}{2 \cdot 4 \cdot 2!} E_1 B_2 x^2 + \frac{2^5-1}{2 \cdot 6 \cdot 4!} E_2 B_3 x^4 + \cdots \quad (a=1) \\
&= \frac{3!}{2} \mu(4) \frac{1}{x^4} - \frac{2^3-1}{2 \cdot 4 \cdot 0!} E_0 B_2 - \frac{2^5-1}{2 \cdot 6 \cdot 2!} E_1 B_3 x^2 - \frac{2^7-1}{2 \cdot 8 \cdot 4!} E_2 B_4 x^4 - \cdots \quad (a=3) \\
&= \frac{5!}{2} \mu(6) \frac{1}{x^6} + \frac{2^5-1}{2 \cdot 6 \cdot 0!} E_0 B_3 + \frac{2^7-1}{2 \cdot 8 \cdot 2!} E_1 B_4 x^2 + \frac{2^9-1}{2 \cdot 10 \cdot 4!} E_2 B_5 x^4 + \cdots \quad (a=5) \\
&\vdots \\
&= \frac{1}{4} \left(\frac{\pi}{2} \right) E_0 \frac{1}{x} \quad (a=0) \\
&= \frac{1}{4} \left(\frac{\pi}{2} \right)^3 E_1 \frac{1}{x^3} \quad (a=2) \\
&= \frac{1}{4} \left(\frac{\pi}{2} \right)^5 E_2 \frac{1}{x^5} \quad (a=4) \\
&\vdots \\
&= \frac{(2^2-1)\pi^2}{2 \cdot 2 \cdot 0! \cdot 2!} E_0 B_1 - \frac{1}{2} \left(\frac{2}{\pi} \right) \mu(2)x \quad (a=-2) \\
&= \frac{(2^4-1)\pi^4}{2 \cdot 2 \cdot 0! \cdot 4!} E_0 B_2 - \frac{(2^2-1)\pi^2}{2 \cdot 2 \cdot 2! \cdot 2!} E_1 B_1 x^2 + \frac{1}{2} \left(\frac{2}{\pi} \right)^3 \mu(4)x^3 \quad (a=-4) \\
&= \frac{(2^6-1)\pi^6}{2 \cdot 2 \cdot 0! \cdot 6!} E_0 B_3 - \frac{(2^4-1)\pi^4}{2 \cdot 2 \cdot 2! \cdot 4!} E_1 B_2 x^2 + \frac{(2^2-1)\pi^2}{2 \cdot 2 \cdot 4! \cdot 2!} E_2 B_1 x^4 - \frac{1}{2} \left(\frac{2}{\pi} \right)^5 \mu(6)x^5 \quad (a=-6) \\
&\vdots
\end{aligned} \tag{89}$$

$$\begin{aligned}
& \frac{1}{e^x + e^{-x}} - \frac{3^a}{e^{3x} + e^{-3x}} + \frac{5^a}{e^{5x} + e^{-5x}} - \frac{7^a}{e^{7x} + e^{-7x}} + \frac{9^a}{e^{9x} + e^{-9x}} - \dots \\
&= \frac{E_0 E_0}{4 \cdot 0!} + \frac{E_1 E_1}{4 \cdot 2!} x^2 + \frac{E_2 E_2}{4 \cdot 4!} x^4 + \frac{E_3 E_3}{4 \cdot 6!} x^6 + \dots \quad (a = 0) \\
&= -\frac{E_0 E_1}{4 \cdot 0!} - \frac{E_1 E_2}{4 \cdot 2!} x^2 - \frac{E_2 E_3}{4 \cdot 4!} x^4 - \frac{E_3 E_4}{4 \cdot 6!} x^6 - \dots \quad (a = 2) \\
&= \frac{E_0 E_2}{4 \cdot 0!} + \frac{E_1 E_3}{4 \cdot 2!} x^2 + \frac{E_2 E_4}{4 \cdot 4!} x^4 + \frac{E_3 E_5}{4 \cdot 6!} x^6 + \dots \quad (a = 4) \\
&\vdots \\
&= \frac{\left(\frac{\pi}{2}\right) E_0 E_0}{2 \cdot 2 \cdot 0! \cdot 0!} \quad (a = -1) \\
&= \frac{\left(\frac{\pi}{2}\right)^3 E_0 E_1}{2 \cdot 2 \cdot 0! \cdot 2!} - \frac{\left(\frac{\pi}{2}\right) E_1 E_0}{2 \cdot 2 \cdot 2! \cdot 0!} x^2 \quad (a = -1) \\
&= \frac{\left(\frac{\pi}{2}\right)^5 E_0 E_2}{2 \cdot 2 \cdot 0! \cdot 4!} - \frac{\left(\frac{\pi}{2}\right)^3 E_1 E_1}{2 \cdot 2 \cdot 2! \cdot 2!} x^2 + \frac{\left(\frac{\pi}{2}\right) E_2 E_0}{2 \cdot 2 \cdot 4! \cdot 0!} x^4 \quad (a = -5) \\
&= \frac{\left(\frac{\pi}{2}\right)^7 E_0 E_3}{2 \cdot 2 \cdot 0! \cdot 6!} - \frac{\left(\frac{\pi}{2}\right)^5 E_1 E_2}{2 \cdot 2 \cdot 2! \cdot 4!} x^2 + \frac{\left(\frac{\pi}{2}\right)^3 E_2 E_1}{2 \cdot 2 \cdot 4! \cdot 2!} x^4 - \frac{\left(\frac{\pi}{2}\right) E_3 E_0}{2 \cdot 2 \cdot 6! \cdot 0!} x^6 \quad (a = -7) \\
&\vdots \\
&= \frac{E_0}{2 \cdot 0!} \mu(2) - \frac{E_1 E_0}{4 \cdot 2!} x^2 - \frac{E_2 E_1}{4 \cdot 4!} x^4 - \frac{E_3 E_2}{4 \cdot 6!} x^6 - \dots \quad (a = -2) \\
&= \frac{E_0}{2 \cdot 0!} \mu(4) - \frac{E_1}{2 \cdot 2!} \mu(2) x^2 + \frac{E_2 E_0}{4 \cdot 4!} x^4 + \frac{E_3 E_1}{4 \cdot 6!} x^6 + \dots \quad (a = -4) \\
&= \frac{E_0}{2 \cdot 0!} \mu(6) - \frac{E_1}{2 \cdot 2!} \mu(4) x^2 + \frac{E_2}{2 \cdot 4!} \mu(2) x^4 - \frac{E_3 E_0}{4 \cdot 6!} x^6 - \dots \quad (a = -6) \\
&\vdots \\
&= 0 \quad (a = 1, 3, 5, 7, \dots)
\end{aligned} \tag{90}$$